ON CHARACTER OF THE PROGRAMMED ITERATION
METHOD CONVERGENCE FOR CONTROL PROBLEMS
WITH ELEMENTS OF UNCERTAINTY*

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Abstract. The article is devoted to the problem of control under uncertainty. The
versions of Programmed Iteration Method are under consideration. The conditions for
Hausdorff convergence of iteration procedure is established. The analogs of Krasovskii-
Subbotin extremal shift rule is considered also.

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1. Introduction. This article is devoted to Programmed Iteration
Method (PIM). This method was used earlier in the theory of differ-
ential games to construct cost functions and stable bridges in sense of
N.N.Krasovskii. We remind that the theory of differential games is a part
of the control theory. The control problems complicated with action of dis-
turbance or uncertainly are considered in the differential game theory. The
structure of nonlinear differential game is characterized exhaustively by the
alternative theorem proved by N.N.Krasovskii and A.I.Subbotin [1], [2]. The
existence of saddle point in the class of positional strategies under conditions
of an information consistency follows from this theorem (see [1]–[7]). Con-
crete construction of cost function and the stable bridges under the conditions
of a regularity (see. [1], [3], [4]) is realized on the basis of auxiliary program

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21
technique is game character of used program control problems. These methods have been developed in N.N.Krasovskii's and his school's works (see [4], [8]–[10]). They have found wide application in the theory of regular differential games (see [1], [4]). The first versions of PIM were constructed on the basis of these techniques [11], [12]; see also works [13]–[15]. The programmed iteration method did not demand the conditions of regularity. These methods also are game: one of participants choice the program (it is a set of a special kind), and another participant choice an answer control. For some classes of irregular differential the "fast" stabilization of iterative sequence is established games (see [7], [11], [12]). The programmed iteration method degenerated to the final procedure in these cases. The expansion constructions were used actively in the theory of programmed iteration method. This technique was applied early for regular differential games. The generalized controls were defined as Borel measures on the Cartesian products of finite dimensional compacts; [7], [11], [12]. The traditional methods of modern measure theory like identification of Borel measures and linear continuous functionals going back to Rietz theorem [18], and representation based on slight regimes [19], [20] are used. The game versions of slight regimes are described in [16]–[21]. The ideas connected to expansions, understood already in the other sense, have found the reflection at A.I.Subbotin's works [21], [22]. These research are devoted to the generalized solution of Hamilton-Jacobi equations. The analogs of programmed iteration method were found an application in mentioned generalized solution construction [23].

The PIM was suggested in several versions. It is known the version for cost function construction, some variants for stable bridge construction [24] and the direct version of programmed iteration method [1], [3]–[8], [7], [12], [23], [24]. The duality of different versions of PIM is considered in [34], [35]. This article is devoted to the early versions of programmed iteration method. The problem of stable bridge construction is under consideration. We are interesting in the character of iteration sequence convergence.

The sense of concerned problem is as follows. We consider a conflict control system on a bounded time interval. The useful control \( u(t) \) and disturbance \( v(t) \) act on this system. The control \( u(t) \) is formed on purpose to the system approach a target set \( M \). This approach is to be realized under the all possible disturbance. The set of this problem successful solvability is a main goal of our constructions.

2. General notions and designations. We use the standard set-theoretical symbolics. In the following \( \Delta \) is the equality by definition. We call by a family a set all elements of which are sets too. Denote by \( \mathcal{P}(X) \) (by
the family of all (all nonempty) subsets of set $X$. If $\mathcal{X}$ is a family and $Y$ is a set, then

$$\mathcal{X}|_Y \triangleq \{ X \cap Y : X \in \mathcal{X} \} \in \mathcal{P}(\mathcal{P}(Y)).$$

Moreover, if $A$ and $B$ are sets, $g$ is a function from $A$ into $B$ and $C \in \mathcal{P}'(A)$, then

$$(g|C) \triangleq (g(a))_{a \in C}$$

is a function from $C$ into $B$.

If $(X, \tau)$, $X \neq \emptyset$, is a metric space, then by $(\text{comp})[X; \tau]$ denote the family of all nonempty compact subsets of $X$ in the sense of the topology generated by the metric $\tau$. For a set $A \in (\text{comp})[X; \tau]$ and a point $x \in X$, define

$$(r - \min)[x; A] \triangleq \min_{a \in A} \tau(x, a) \in [0, \infty[.$$

So we introduce the distance from $x$ to the set $A$. Moreover, we introduce the Hausdorff metric

$$H_{\tau} : (\text{comp})[X; \tau] \times (\text{comp})[X; \tau] \to [0, \infty[$$

by the natural rule: for any $A \in (\text{comp})[X; \tau]$ and $B \in (\text{comp})[X; \tau]$

$$H_{\tau}(A, B) \triangleq \sup \{ \max_{x \in A} (r - \min)[x; B]; \max_{x \in B} (r - \min)[x; A] \}. \quad (1)$$

2.1. Elements of measure theory. For a measurable space $(MS)(X, \mathcal{X})$ ($X$ is a set and $\mathcal{X}$ is a $\sigma$-algebra of subsets of $X$), we denote by $(\sigma - \text{add})[\mathcal{X}]$ (by $(\sigma - \text{add})_+[\mathcal{X}]$) the set of all (all nonnegative) real-valued measures on the $\sigma$-algebra $\mathcal{X}$. As usual $\mathcal{X}$ is a $\sigma$-algebra of Borel sets in a metric space. In this case any measures of $(\sigma - \text{add})_+[\mathcal{X}]$ are regular. Of course, we use Rietz Theorem about the representation of a space of linear functionals on a space of continuous functions.

Recall one useful statement of general topology; see, for example [28, corollary 3.1.5]. Namely, if $(X, \tau)$ is a topological space, $U \in \tau$, and $\mathcal{F}$ is a family of closed (in $(X, \tau)$) subsets of $X$ for which at least one set of $\mathcal{F}$ is compact and

$$\bigcap_{F \in \mathcal{F}} F \subset U,$$

(2)
then there exists a nonempty finite family $\mathcal{Y}$, $\mathcal{Y} \subset \mathcal{F}$, for which

$$\bigcap_{F \in \mathcal{Y}} F \subset U.$$  \hspace{1cm} (3)

In particular, if $\mathcal{F}$ is a nonempty family of compact sets in a Hausdorff space $(X, \tau)$, then from (2), the inclusion (3) follows for a finite $\mathcal{Y} \in \mathcal{P}'(\mathcal{F})$.

Suppose that $\mathbb{R}$ is the real line; all intervals in $\mathbb{R}$ are denoted by square brackets only. Let $\mathcal{N} \triangleq \{1; 2; 3; \ldots\}$ and $\mathcal{N}_0 \triangleq \{0\} \cup \mathcal{N} = \{0; 1; 2; 3; \ldots\}$. If $m \in \mathcal{N}$, then $\overline{m, \infty} \triangleq \{i \in \mathcal{N} | m \leq i\}$. Of course, we suppose that any natural numbers $k \in \mathcal{N}$ are not set. If $U$ is a set and $m \in \mathcal{N}$, then $U^m$ denotes (as usual) the set of processions

$$(u_j)_{j \in \overline{1, m}} : \overline{1, m} \rightarrow U.$$  

In particular, for $m \in \mathcal{N}$, $\mathbb{R}^m$ is $m$-dimensional arithmetic space equipped with the Euclidean metric $\rho_m$ generated by corresponding norm.

2.2. The conflict controlled system: generalized controls and solutions. We consider control process on the finite time intervals

$I_0 \triangleq [t_0, \theta_0],

$$(\text{here } t_0 \in \mathbb{R} \text{ and } \theta_0 \in ]t_0, \infty[) \text{ and on the time intervals}$

$[t, \theta_0], \ t \in I_0.$

For $t \in I_0$, we equip the set $[t, \theta_0]$ with the Borel $\sigma$-algebra $\mathcal{T}_t$ (in particular $\mathcal{T}_0$ is the $\sigma$-algebra generated by the semialgebra of all intervals contained in $I_0$; of course we keep in mind open, closed, and semiclosed intervals $I$, $I \subset I_0$). We fix $n \in \mathcal{N}$, $p \in \mathcal{N}$ and $q \in \mathcal{N}$. Moreover, we fix

$$P \in (\text{comp})[\mathbb{R}^p; \rho_p], \ Q \in (\text{comp})[\mathbb{R}^q; \rho_q].$$

We consider the compact sets $P$ and $Q$ as the sets of instantaneous controls of the players I and II respectively. Moreover, we fix the function

$$f : I_0 \times \mathbb{R}^n \times P \times Q \rightarrow \mathbb{R}^n.$$  

As a result we obtain the conflict control system

$$\dot{x} = f(t, x, u, v), \ u \in P, \ v \in Q.$$  \hspace{1cm} (4)
We suppose that the function $f$ satisfies the traditional conditions for the differential games theory problems. Namely let $f$ is a continuous function. Moreover $f$ is locally lipschitzian function with respect to the phase variable $x \in \mathbb{R}^n$ [1]: of course we keep in mind the functions $f(t, \cdot, u, v), t \in I_0, u \in P, v \in Q$. Finally, we suppose that there exist a number $\kappa \in ]0, \infty[$ such that
\[
\|f(t, x, u, v)\| \leq \kappa(1 + \|x\|) \forall t \in I_0 \forall x \in \mathbb{R}^n \forall u \in P \forall v \in Q; \tag{5}
\]
here and below $\| \cdot \|$ is the Euclidian norm in $\mathbb{R}^n$.

We use sliding regime for the system (4). Therefore, we introduce some special constructions corresponds to [7], [12]. In the investigations the so-called strategical measure are used. We keep in mind measures on Cartesian products; suppose that for $t \in I_0$, 
\[
(Y_t \triangleq [t, \vartheta_0] \times P) \&(Z_t \triangleq [t, \vartheta_0] \times Q) \&(\Omega_t \triangleq [t, \vartheta_0] \times P \times Q). \tag{6}
\]
In (6) we have metrizable compactums. For any $t \in I_0$ we introduce the Borel $\sigma$-algebras $\mathcal{K}_t, \mathcal{D}_t,$ and $\mathcal{C}_t$ of Borel subsets of $Y_t, Z_t,$ and $\Omega_t$ respectively; now we obtain the measurable spaces 
\[
([t, \vartheta_0], \mathcal{T}_t), \ (Y_t, \mathcal{K}_t), \ (Z_t, \mathcal{D}_t), \ (\Omega_t, \mathcal{C}_t)
\]
and suppose that 
\[
\lambda_t \in (\sigma - \text{add})_+[\mathcal{T}_t]
\]
is the Lebesque-Borel measure on $[t, \vartheta_0]$. For $t \in I_0$ usual controls on $[t, \vartheta_0]$ are defined as Borel functions 
\[
u(\cdot) : [t, \vartheta_0] \rightarrow P, \quad v(\cdot) : [t, \vartheta_0] \rightarrow Q. \tag{7}
\]
Moreover, we consider pairs $(u(\cdot), v(\cdot))$ such functions as common controls in the system (4).

In the general case the generalized analogs of usual controls (7) are needed. Let us remark that 
\[
(\Gamma \times P \in \mathcal{K}_t) \&(\Gamma \times Q \in \mathcal{D}_t) \&(\Gamma \times P \times Q \in \mathcal{C}_t) \tag{8}
\]
for $t \in I_0$ and $\Gamma \in \mathcal{T}_t$.

If $t \in I_0$ and $D \in \mathcal{D}_t$ then 
\[
D \otimes P \triangleq \{(\tau, u, v) \in \Omega_t | (\tau, v) \in D \} \in \mathcal{C}_t. \tag{9}
\]
The relation (9) is the development of the last statement in (8).

Now we introduce the following three sets

\[ R_t \triangleq \{ \mu \in (\sigma - \text{add})_+[\mathcal{K}_t] | \mu(\Gamma \times P) = \lambda_t(\Gamma) \ \forall \Gamma \in \mathcal{T}_t \}, \]

\[ E_t \triangleq \{ \nu \in (\sigma - \text{add})_+[\mathcal{D}_t] | \nu(\Gamma \times Q) = \lambda_t(\Gamma) \ \forall \Gamma \in \mathcal{T}_t \}, \]

\[ H_t \triangleq \{ \eta \in (\sigma - \text{add})_+[\mathcal{C}_t] | \eta(\Gamma \times P \times Q) = \lambda_t(\Gamma) \ \forall \Gamma \in \mathcal{T}_t \}. \]

Measures \( \mu \in R_t \) and \( \nu \in E_t \) are generalized analogs of usual controls \( u(\cdot) \) and \( v(\cdot) \) in (7) respectively; measures \( \eta \in H_t \) are generalized analogs of the pairs \( (u(\cdot), v(\cdot)) \).

For \( t \in I_0 \) and \( \nu \in E_t \) define the set

\[ \Pi_t[\nu] \triangleq \{ \eta \in H_t | \eta(D \otimes P) = \nu(D) \ \forall D \in \mathcal{D}_t \}. \]

The set (10) is the generalized analog of the set of all pairs \( (u(\cdot), v(\cdot)) \) of usual controls, if \( v(\cdot) = \vartheta(\cdot) \), where \( \vartheta(\cdot) \) is a fixed Borel control with values in \( Q \).

Further we use two schemes of formation of \( \eta \in H_t \). In the first scheme a hypothetical player II chooses a "control" \( \nu \in E_t \) and then a "player" I chooses a measure \( \eta \in \Pi_t[\nu] \). The corresponding effect of the choice of \( \eta \) is showed later. The second scheme is realized as follows: the "player" II chooses \( \nu \in Q \) and the "player" I chooses \( \mu \in R_t \). As a result some general control \( \eta \in H_t \) is realized.

We note that for \( t \in I_0 \) all measures owned to the sets \( R_t, E_t, H_t \) are regular; see for example [17]. Therefore, the action of these measures come to the action of linear continuous functionals on the spaces \( C(Y_t), C(Z_t) \), and \( C(\Omega_t) \) of continuous functions on \( Y_t, Z_t \), and \( \Omega_t \) respectively; \( t \in I_0 \). It follows from well-known Rietz theorem. Therefore \( R_t, E_t, \) and \( H_t \) are identified with subsets of the spaces \( C^*(Y_t), C^*(Z_t), \) and \( C^*(\Omega_t) \) topologically conjugated with \( C(Y_t), C(Z_t), \) and \( C(\Omega_t) \) respectively. In addition under \( t \in I_0 \)

\[ H_t = \bigsqcup_{\nu \in E_t} \Pi_t[\nu], \]

where \( \bigsqcup \) denotes the disjunctive union of the corresponding sets; see [29].

In the following we use sliding regimes. The basic type of these sliding regimes corresponds to the employment of measures \( \eta \in H_t (t \in I_0) \). Also,
we use a variant of sliding regimes for so-called v-systems (see [29] and [7, P. 232]). In this variant, we consider trajectories generated by measures \( \mu \in \mathcal{R}_t, t \in I_0 \).

Now let us consider the basic type of sliding regimes. If \( t_* \in I_0, x_* \in \mathbb{R}^n \), and \( \eta \in \mathcal{H}_{t_*} \), then denote by

\[
\varphi(\cdot, t_*, x_*, \eta) = (\varphi(t, t_*, x_*, \eta))_{t \in [t_*, \vartheta_0]} (11)
\]

the unique continuous function from \([t_*, \vartheta_0]\) into \( \mathbb{R}^n \) for which

\[
\varphi(t, t_*, x_*, \eta) = x_* + \int_{[t_*, t_0 \times P \times Q} f(\tau, \varphi(\tau, t_*, x_*, \eta), u, v) \eta(d(\tau, u, v)) \forall t \in [t_*, \vartheta_0]; (12)
\]

(see [7], [11], [12], [29], [30]). We have the generalized solutions similar to [1], [5], [19], [20] in (11) and (12). Namely it is possible to consider measurable mappings

\[
t \mapsto \eta_t : [t_*, \vartheta_0] \rightarrow \{ \mu \in (\sigma - \text{add})_+[\mathfrak{P} \otimes \Omega] | \mu(P \times Q) = 1 \}, (13)
\]

here \( \mathfrak{P} \) and \( \Omega \) are the \( \sigma \)-algebras of Borel sets of \( P \) and \( Q \) respectively; \( \mathfrak{P} \times \Omega \) is the standard \( \sigma \)-algebra of subsets of the product of measurable spaces \((P, \mathfrak{P})\) and \((Q, \Omega)\); see [17]. For these measure-valued functions (13) one may consider the differential equations

\[
\dot{x}(t) = \int_{P \times Q} f(t, x(t), u, v) \eta_t(d(u, v)) \text{ almost everywhere} (14)
\]

\( x(t_*) = x_* \). Of course the solution of (14) is understood in the sense of Karatheodory. Then the equations (12) may be considered as the integral forms of (14).

Now we consider so-called v-system [29]

\[
\dot{x} = f_v(t, x, u), u \in P, \quad (15)
\]

here \( f_v : I_0 \times \mathbb{R}^n \times P \rightarrow \mathbb{R}^n \) is defined by the formula:

\[
f_v(t, x, u) \triangleq f(t, x, u, v) \forall t \in I_0 \forall x \in \mathbb{R}^n \forall u \in P. (16)
\]

When we consider the v-system (15), (16) we keep in mind the following scheme of control in (4): the player I chooses a measure \( \mu \in \mathcal{R}_{t_*} \) and the
player II chooses a constant control $v \in Q$. There exist the unique continuous solution

$$\varphi_v(\cdot, t_*, x_*, \mu) = (\varphi_v(t, t_*, x_*, \mu))_{t \in [t_*, \vartheta_0]}$$

of the integral equation

$$\varphi_v(t, t_*, x_*, \mu) = x_* + \int_{[t_*, t \in P]} f(\tau, \varphi_v(\tau, t_*, x_*, \mu), u, v) \mu(d(\tau, u)) \forall t \in [t_*, \vartheta_0]$$

under fixed $t_* \in I_0$ and $x_* \in \mathbb{R}^n$. We have the joint action of $\mu$ and $v$ in (17) and (18). This action may be considered in the next form.

Namely, if $t_* \in I_0$, $\mu_* \in \mathcal{R}_{t_*}$ and $v_* \in Q$ then it is possible to realize the unique measure $\eta_* \in \mathcal{H}_{t_*}$ such that

$$\int_{\Omega_{t_*}} g(t, u, v) \eta_*(d(t, u, v)) = \int_{Y_{t_*}} g(t, u, v_*) \mu_*(d(t, u))$$

for any continuous real-valued functions $g$ on $\Omega_{t_*}$. It follows from well-known Rietz theorem. Recall that $\eta_*$ is denoted as $\mu_* \otimes v_*$ in [29].

If $t \in I_0$ and $\hat{\vartheta} \in Q$, then $\lambda_t \otimes \hat{\vartheta} \in \mathcal{E}_t$ is by definition a unique measure such that

$$\int_{Z_t} g(\tau, v)(\lambda_t \otimes \hat{\vartheta})(d(\tau, v)) = \int_{[t, \vartheta_0]} g(\tau, \hat{\vartheta}) \lambda_t d\tau \forall g \in C(Z_t).$$

One can prove that

$$\mu \otimes v \in \Pi_t[\lambda_t \otimes v]$$

for any $t \in I_0$, $\mu \in \mathcal{R}_t$ and $v \in Q$.

Let us introduce several special notions and designations in view of the multivalued generalized quasistrategies employment. Let $t \in I_0$ and $\theta \in [t, \vartheta_0]$. Note that

$$\mathcal{D}_t|_{[t, \theta] \times Q} = \{ D \cap [t, \theta[ \times Q) : D \in \mathcal{D}_t \} = \{ D \in \mathcal{D}_t | D \subset [t, \theta[ \times Q \} \in \mathcal{P}'(\mathcal{D}_t),$$

$$\mathcal{C}_t|_{[t, \theta] \times P \times Q} = \{ H \cap ([t, \theta[ \times P \times Q) : H \in \mathcal{C}_t \} = \{ H \in \mathcal{C}_t | H \subset [t, \theta[ \times P \times Q \} \in \mathcal{P}'(\mathcal{C}_t).$$

Suppose,

$$\forall \nu \in \mathcal{E}_t \& \forall H \in \mathcal{P}(\mathcal{H}_t)$$

$$([H; \theta] \triangleq \{ (\nu|_{\mathcal{D}_t}|_{[t, \theta] \times Q}) : \nu \in \mathcal{E}_t \} \&$$

$$\{ (\eta|_{\mathcal{C}_t}|_{[t, \theta] \times P \times Q}) : \eta \in \mathcal{H}_t \} \forall H \in \mathcal{P}(\mathcal{H}_t).$$
3. Quasistrategies. In this section we recall some constructions of control in the class of multivalued quasistrategies; [11], [12], [25]. These constructions may be considered as the natural development of an approach used by Roxin [31], Elliot and Kalton [32], Varaja and Lin [33] and other mathematicians. In natural way the question is the nonanticipatory reactions of the form \( u(\cdot) = \alpha(v(\cdot)) \). Here \( u(\cdot) \) and \( v(\cdot) \) are the usual program control on the fixed time interval \([t_*, \theta_0]\), \( t_* \in I_0 \). We use the generalized control that is the measures. It is connected with possible noncompactness of the corresponding bundle of trajectories. The features of our constructions are the used quasistrategies multivalueness and the employment of measure in these constructions. Used in our construction quasistrategies is generalized variants of the mappings

\[
v(\cdot) \mapsto \alpha(v(\cdot)),
\]

where \( \alpha(v(\cdot)) \) is a nonempty set of Borel functions from \([t_*, \theta_0]\) into \( P \) (of course, \( v(\cdot) \) is a Borel function from \([t_*, \theta_0]\) into \( Q \)). In addition we require the nonanticipating property for \( \alpha (19) \).

We consider the corresponding analogs of (19) under the measure spaces.

If \( t \in I_0 \) then by \( A_t \) denote the set of mappings

\[
\alpha \in \prod_{\nu \in \mathcal{E}_t} \mathcal{P}'(\Pi_{t}[\nu])
\]

such that for any \( \nu_1 \in \mathcal{E}_t, \nu_2 \in \mathcal{E}_t \) and \( \theta \in [t, \theta_0] \), the following implication holds (see [11], [12], [7])

\[
([\nu_1; \theta] = [\nu_2; \theta]) \Rightarrow ([\alpha(\nu_1); \theta] = [\alpha(\nu_2); \theta]).
\]

Note that \( A_t \neq \emptyset \). It follows from the next constructions.

**Remark 1.** If \( t \in I_0 \) then \( \nu \mapsto \Pi_t[\nu]: \mathcal{E}_t \rightarrow \mathcal{P}'(\mathcal{H}_t) \) is an elements of \( A_t \). For it we note that \( \Pi_t[\nu] \neq \emptyset \ \forall \nu \in \mathcal{E}_t \); see for example [29, p. 14,15]. For \( \nu_1 \in \mathcal{E}_t \) and \( \nu_2 \in \mathcal{E}_t \) with the property contained in the premise of (21) one can realize the slicing procedure. Namely choose an arbitrary function \( \gamma_1 \in [\Pi_t(\nu_1); \theta >; \)

\[
\gamma_1: C_t|[t, \theta] \times P \times Q \rightarrow [0, \infty[,
\]

and for some \( \eta_1 \in \Pi_t[\nu_1] \)

\[
\gamma_1 = (\eta_1 | C_t|[t, \theta] \times P \times Q).
\]
The last relation means that for any $H \in C_{t}[t, \theta] \times \mathbb{P} \times \mathbb{Q}$

$$\gamma_{1}(H) = \eta_{1}(H).$$

Choose arbitrary $\eta_{2} \in \Pi_{t}[\nu_{2}]$; let us consider the family $D_{\theta}$. It is easy to see that

$$(D_{\theta} \in \mathcal{P}(D_{\theta})) \& (C_{\theta} \in \mathcal{P}(C_{t})).$$

Hence $\bar{\eta} \triangleq (\eta_{2}|C_{\theta}) \in \mathcal{H}_{\theta}$. Furthermore for the measure $\bar{\nu}_{2} \triangleq (\nu_{2}|D_{\theta}) \in \mathcal{E}_{\theta}$ (see, [29, p. 17]) the following properties holds

$$\bar{\eta}_{2}(D \otimes P) = \eta_{2}(D \otimes P) = \nu_{2}(D) = \bar{\nu}_{2}(D) \forall D \in D_{\theta}. \quad (22)$$

So, $\bar{\eta}_{2} \in \Pi_{\theta}(\bar{\nu}_{2})$. Further we have (see [29, p. 17])

$$\eta_{1} \square \bar{\eta}_{2} \triangleq (\eta_{1}(H \cap ([t, \theta] \times \mathbb{P} \times \mathbb{Q})) + \eta_{2}(H \cap \mathbb{Q}))_{H \in C_{t}} \in \mathcal{H}_{t}. \quad (23)$$

And what is more $\eta_{1} \square \bar{\eta}_{2} \in \Pi_{t}(\nu_{2})$. Indeed, if $D \in D_{t}$, then (see (21), 22)

$$(\eta_{1} \square \bar{\eta}_{2})(D \otimes P) = \eta_{1}((D \otimes P) \cap ([t, \theta] \times \mathbb{P} \times \mathbb{Q})) + \eta_{2}((D \otimes P) \cap \mathbb{Q}) =$$

$$= \eta_{1}((D \otimes P) \cap ([t, \theta] \times \mathbb{Q}) \otimes P) + \eta_{2}((D \otimes P) \cap \mathbb{Q} \otimes P) =$$

$$= \nu_{1}(D \cap ([t, \theta] \times \mathbb{Q})) + \nu_{2}(D \cap \mathbb{Q}) = \nu_{2}(D \cap ([t, \theta] \times \mathbb{Q}) \cap \mathbb{Q}) + \nu_{2}(D \cap \mathbb{Q}) = \nu_{2}(D).$$

Since the choice of $D$ was arbitrary, then the required property for $\eta_{1} \square \bar{\eta}_{2}$ is established. Therefore,

$$\gamma_{2} \triangleq (\eta_{1} \square \bar{\eta}_{2}|C_{t}[t, \theta] \times \mathbb{P} \times \mathbb{Q}) \in \Pi_{t}(\nu_{2}); \theta >. \quad (24)$$

But from (23) we obtain that

$$\gamma_{2}(H) = \eta_{1}(H) = \gamma_{1}(H) \forall H \in C_{t}[t, \theta] \times \mathbb{P} \times \mathbb{Q}.$$ 

Consequently, $\gamma_{1} = \gamma_{2}$. Since the choice of $\gamma_{1}$ was arbitrary, too we have (see (24)) the inclusion

$$[\Pi_{t}(\nu_{1}); \theta >] \subset [\Pi_{t}(\nu_{2}); \theta >]$$

The choice of $\nu_{1}$ and $\nu_{2}$ was arbitrary too. So, the required property of the mapping $\Pi_{t}[\cdot]$ is established.
We consider the above-mentioned multivalued quasistrategies as control procedures. In addition, we confine ourself to the consideration of approach-evasion differential game. Let us fix two sets

\[ M \in \mathcal{P}(I_0 \times \mathbb{R}^n), \quad N \in \mathcal{P}(I_0 \times \mathbb{R}^n). \]

Suppose that \( M \) and \( N \) are the close sets in \( I_0 \times \mathbb{R}^n \) with the usual topology of the coordinatewise convergence (the more general case was considered in [29]). Let us introduce the set \( \mathcal{W} \) of all positions of \( N \) such that there exist a quasistrategy \( \alpha \in A_t \) for which

\[ \forall \eta \in \bigcup_{\nu \in \varepsilon_1} \alpha(\nu) \exists \theta \in [t, \theta_0] : \]

\[ ((\theta, \varphi(\theta, t, t, \eta, x)) \in M) \land ((\xi, \varphi(\xi, t, x, \eta)) \in N \forall \xi \in [t, \theta]). \]

Then, \( \mathcal{W} \) is the positional absorption set in sense of N. N. Krasovskii and A. I. Subbotin concurrently; see [29], [11] in this connection.

We consider \( \mathcal{W} \) as the solution of the problem about the partition of the position space into the Krasovskii-Subbotin alternative partition.

**Remark 2.** Under our condition for choice of \( M \) and \( N \),

\[ \mathcal{W} = \{(t, x) \in N | \exists \alpha \in A_t \forall \eta \in \bigcup_{\nu \in \varepsilon_1} \alpha(\nu) \exists \theta \in [t, \theta_0] : \]

\[ ((\theta, \varphi(\theta, t, t, \eta, x)) \in M) \land ((\xi, \varphi(\xi, t, x, \eta)) \in N \forall \xi \in [t, \theta]) \} \]  

(25)

Note that in the set of right hand of (25), the requirement of the belonging to \( N \) is amplified. Therefore the set \( \mathcal{W}_0 \) on the right hand of (25) is a subset of \( \mathcal{W} \). Let \( (t_*, x_*) \in \mathcal{W} \). Then \( (t_*, x_*) \in N \) and for some \( \alpha_* \in A_t \) we have the property:

\[ \forall \eta \in \bigcup_{\nu \in \varepsilon_1} \alpha_*(\nu) \exists \theta \in [t_*, \theta_0] : ((\theta, \varphi(\theta, t_*, t_*, \eta, x_*)) \in M) \land \]

\[ ((\xi, \varphi(\xi, t_*, x_*, \eta)) \in N \forall \xi \in [t_*, \theta]). \]

Let

\[ \eta_* \in \bigcup_{\nu \in \varepsilon_1} \alpha_*(\nu). \]

Then, for some \( \theta_* \in [t_*, \theta_0] \)

\[ ((\theta_*, \varphi(\theta_*, t_*, x_*, \eta_*)) \in M) \land ((\xi, \varphi(\xi, t_*, x_*, \eta_*)) \in N \forall \xi \in [t_*, \theta_*]). \]  

(26)

If \( [t_*, \theta_*] \neq \emptyset \), then \( (\theta_*, \varphi(\theta_*, t_*, x_*, \eta_*)) \in N \) due to the continuity of \( \varphi(\cdot, t_*, x_*, \eta_*) \) (since \( N \) is a closed set); as a corollary the last statement in (26) is amplified \( (\xi, \varphi(\xi, t_*, x_*, \eta_*)) \in N \) for any \( \xi \in [t_*, \theta_*] \).
Let \([t_*, \theta_*] = \emptyset\). Then \(t_* = \theta_*\), \([t_*, \theta_*] = \{t_*\}\), and \((t_*, \varphi(t_*, t_*, x_*, \eta_*)) = (t_*, x_*) \in N\). Therefore in this (obvious) case we obtain that \((\xi, \varphi(\xi, t_*, x_*, \eta_*)) \in N\) for any \(\xi \in [t_*, \theta_*]\).

Connecting above-mentioned cases we have that

\[ (\xi, \varphi(\xi, t_*, x_*, \eta_*)) \in N \forall \xi \in [t_*, \theta_*] \]

always. Since the choice of \(\eta_*\) was arbitrary, we obtain that

\[ \forall \eta \in \cup_{\nu \in E_{t_*}} \alpha_*(\nu) \exists \theta \in [t_*, \theta_0]: \]

\[ (((\theta, \varphi(\theta, t_*, x_*, \eta))) \in M) \& ((\xi, \varphi(\xi, t_*, x_*, \eta)) \in N \forall \xi \in [t_*, \theta])) \]

Therefore, \((t_*, x_*) \in W_0\) is established. As a result, we obtain the equality (25).

In connection with (25) we note else one equality:

\[ W = \{(t, x) \in I_0 \times \mathbb{R}^n | \exists \alpha \in A_{t_*} \forall \eta \in \cup_{\nu \in E_t} \alpha(\nu) \exists \theta \in [t_*, \theta_0]: \]

\[ (((\theta, \varphi(\theta, t, x, \eta))) \in M) \& ((\xi, \varphi(\xi, t, x, \eta)) \in N \forall \xi \in [t, \theta])) \} \quad (27) \]

Indeed, for the set \(W\) (25) and \(W^0\) on the right hand of (27) we have the obvious inclusion

\[ W \subset W^0. \]

If \((t^*, x^*) \in W^0\), then \((t^*, x^*) \in I_0 \times \mathbb{R}^n\) and for some \(\alpha^* \in A_{t^*}\) the following property holds

\[ \forall \eta \in \cup_{\nu \in E_{t^*}} \alpha^*(\nu) \exists \theta \in [t^*, \theta_0]: \]

\[ (((\theta, \varphi(\theta, t^*, x^*, \eta))) \in M) \& ((\xi, \varphi(\xi, t^*, x^*, \eta)) \in N \forall \xi \in [t^*, \theta]) \quad (28) \]

By (20) the union of all sets \(\alpha^*(\nu), \nu \in E_{t^*}\) is a nonempty set. Choose \(\eta^* \in \cup_{\nu \in E_{t^*}} \alpha^*(\nu)\). Then \(\eta^* \in H_{t^*}\) and for a \(\theta^* \in [t^*, \theta_0]\)

\[ (((\theta^*, \varphi(\theta, t^*, x^*, \eta^*))) \in M) \& ((\xi, \varphi(\xi, t^*, x^*, \eta^*)) \in N \forall \xi \in [t^*, \theta^*]) \]. \quad (29) \]

From (29) we have in particular the inclusion \((t^*, x^*) = (t^*, \varphi(t^*, t^*, x^*, \eta^*)) \in N\). Then by (25) and (28) the inclusion \((t^*, x^*) \in W\) follows. Consequently, \(W \subset W^0\). Therefore, (27) is valid.
The above-mentioned reasonings remain true if we substitute close set 
\( F \subseteq N \) for \( N \).

\[
\{(t, x) \in F | \exists \alpha \in A_t \forall \eta \in \cup_{\nu \in E_t} \alpha(\nu) \exists \theta \in [t, \theta_0] : \\
((\theta, \varphi(\theta, t, x, \eta)) \in M) \& ((\xi, \varphi(\xi, t, x, \eta)) \in F \forall \xi \in [t, \theta]) \} = \\
= \{(t, x) \in F | \exists \alpha \in A_t \forall \eta \in \cup_{\nu \in E_t} \alpha(\nu) \exists \theta \in [t, \theta_0] : \\
((\theta, \varphi(\theta, t, x, \eta)) \in M) \& ((\xi, \varphi(\xi, t, x, \eta)) \in F \forall \xi \in [t, \theta]) \} = \\
= \{(t, x) \in I_0 \times \mathbb{R}^n | \exists \alpha \in A_t \forall \eta \in \cup_{\nu \in E_t} \alpha(\nu) \exists \theta \in [t, \theta_0] : \\
((\theta, \varphi(\theta, t, x, \eta)) \in M) \& ((\xi, \varphi(\xi, t, x, \eta)) \in F \forall \xi \in [t, \theta]) \}
\]

Now we may use statements in [11], [24]; moreover see Theorem 4.4.3 in [7]. In particular, the last theorem reduced to (25).

Let us show the quasistrategies solved the approach problem. For a closed set \( F, F \subseteq N \), a position \((t_*, x_*) \in I_0 \times \mathbb{R}^n\), and \( \nu \in E_{t_*} \) define

\[
\pi_{t_*,x_*}(\nu|F) \triangleq \{ \eta \in \Pi_{t_*}[\nu] | \exists \theta \in [t_*, \theta_0] : ((\theta, \varphi(\theta, t_*, x_*, \eta)) \in M) \& ((\xi, \varphi(\xi, t_*, x_*, \eta)) \in F \forall \xi \in [t_*, \theta]) \}. \tag{30}
\]

Moreover for a closed set \( F, F \subseteq N \), and a position \((t_*, x_*) \in I_0 \times \mathbb{R}^n\) we suppose

\[
\pi_{t_*,x_*}(.|F) \triangleq (\pi_{t_*,x_*}(\nu|F)_{\nu \in E_{t_*}}).
\]

It is follows from [7, Theorem 4.4.1], [11], and [24] that if \((t, x) \in \mathcal{W}\) then

\[
\pi_{t,x}(.|\mathcal{W}) \in A_t
\]

and

\[
\forall \eta \in \cup_{\nu \in E_t} \pi_{t,x}(\nu|\mathcal{W}) \exists \theta \in [t, \theta_0] : ((\theta, \varphi(\theta, t, x, \eta)) \in M) \& ((\xi, \varphi(\xi, t, x, \eta)) \in N \forall \xi \in [t, \theta]).
\]

So, the structure of quasistrategy solved the approach problem is known for positions from \( \mathcal{W} \). This quasistrategy is defined (see 30) as solution of the programmed problem approach onto \( M \) under phase constraints defined by \( \mathcal{W} \).

So, in principle the solving of problem under consideration is reduced to constructing the set \( \mathcal{W} \). For this aim we use well-known PIM [7], [29], [11], [25]. For brevity we use the term PIM. Of course many variants of PIM are
known. Each variant of PIM is characterized by some programmed absorption operator. In this connection we note [34] [35]. But, now we consider the earlier variants of PIM; see [29], [11], [24]. Now, we consider only iteration procedures in the spaces of sets. We consider conditions guaranteed the realization of sets in the family of compacts. Finally we investigate conditions of the convergence in the Hausdorff metric (we keep in mind the convergence of iteration sequence to the set \( \mathcal{M} \)). For such procedure realization we use one simple property connected with Gronwall lemma; see [19]

For any \( t_0 \in I_0, \vartheta \in [t_*, \vartheta_0], \gamma \in ]0, \infty[, L \in ]0, \infty[, \alpha \in [0, \infty[ \) and a continuous real-valued function \( h \) on \([t_*, \vartheta]\) the following implication is valid:

\[
(h(t) \leq \alpha + \int_t^\vartheta (\gamma + Lh(\xi))d\xi \forall t \in [t_*, \vartheta]) \Rightarrow \\
(h(t) \leq \frac{\gamma}{L}(e^{L(\vartheta-t)} - 1) + \alpha e^{L(\vartheta-t)} \forall t \in [t_*, \vartheta]). \quad (31)
\]

From (31), the next property of generalized programmed motion follows. If \( t_0 \in I_0, \vartheta \in [t_*, \vartheta_0] \), then for \( \varphi_\eta = \varphi(\cdot, t_*, x_*, \eta) \) and \( t \in [t_*, \vartheta] \)

\[
\varphi_\eta(\vartheta) = \varphi_\eta(t) + \int_{[t, \vartheta] \times P \times Q} f(\xi, \varphi_\eta(\xi), u, v)\eta(d(\xi, u, v)),
\]

and by triangle inequality

\[
\|\varphi_\eta(t)\| = \|\varphi_\eta(\vartheta)\| + \int_{[t, \vartheta] \times P \times Q} \|f(\xi, \varphi_\eta(\xi), u, v)\|\eta(d(\xi, u, v)). \quad (32)
\]

By (5) and (32) we obtain that

\[
\|\varphi_\eta(t)\| \leq \|\varphi_\eta(\vartheta)\| + \int_{[t, \vartheta] \times P \times Q} \alpha(1 + \|\varphi_\eta(\xi)\|)\eta(d(\xi, u, v)) = \\
= \|\varphi_\eta(\vartheta)\| + \int_t^\vartheta \alpha(1 + \|\varphi_\eta(\xi)\|)d\xi.
\]

With the employment of (31) we obtain for any \( t_0 \in I_0, \vartheta \in [t_*, \vartheta_0] \), \( x_* \in \mathbb{R}^n \), and \( \eta \in \mathcal{H}_{t_*} \)

\[
\|\varphi(t, t_*, x_*, \eta)\| \leq (e^{\alpha(\vartheta-t)} - 1) + \|\varphi(\vartheta, t_*, x_*, \eta)\|e^{\alpha(\vartheta-t)} \forall t \in [t_*, \vartheta]. \quad (33)
\]

In particular it is follows from (33) for any \( t_0 \in I_0, \vartheta \in [t_*, \vartheta_0] \), \( x_* \in \mathbb{R}^n \), and \( \eta \in \mathcal{H}_{t_*} \)

\[
\|x_*\| \leq (e^{\alpha(\vartheta-t_*)} - 1) + \|\varphi(\vartheta, t_*, x_*, \eta)\|e^{\alpha(\vartheta-t_*)}. \quad (34)
\]
CONVERGENCE OF PROGRAMMED ITERATION METHOD

Let us introduce the metric on a space $I_0 \times \mathbb{R}^n$. If $t_1 \in I_0$, $x \in \mathbb{R}^n$, $t_2 \in I_0$, and $x_2 \in \mathbb{R}^n$ define

$$d((t_1, x_1), (t_2, x_2)) \triangleq \sup \{|t_1 - t_2|; \|x_1 - x_2\|\}. \quad (35)$$

Of course, the convergence of sequences in the sense of (35) is equivalent to the coordinate-wise convergence in $I_0 \times \mathbb{R}^n$.

Further we suggest the following condition:

$$M \cap N \in \text{(comp)}[I_0 \times \mathbb{R}^n; d].$$

From (25) we have the next simple representation

$$\mathcal{W} = \{(t, x) \in N | \exists \alpha \in A_t \forall \eta \in \cup_{\nu \in \epsilon_t} \alpha(\nu) \exists \theta \in [t, \theta_0] : ((\theta, \varphi(\theta, t, x, \eta)) \in M \cap N) \& ((\xi, \varphi(\xi, t, x, \eta)) \in N \forall \xi \in [t, \theta])\}. \quad (36)$$

For any $(t, x) \in I_0 \times \mathbb{R}^n$, denote

$$d(t, x) = \sup \{|t - t_0; \|x\|\} \in [0, \infty]$$

Due to the set $M \cap N$ compactness the boundness property of the function $(d|M \cap N)$ is valid. Namely, for some $a \in [0, \infty[$

$$d(t, x) \leq a \forall (t, x) \in M \cap N \quad (37)$$

**Remark 3.** From (37) the next property of $\mathcal{W}$ follows:

$$\exists c \in [0, \infty[ : d((t_1, x_1), (t_2, x_2)) \leq c \forall (t_1, x_1) \in \mathcal{W} \forall (t_2, x_2) \in \mathcal{W}. \quad (38)$$

Indeed in the case $\mathcal{W} = \emptyset$ (38) is valid obviously. Denote

$$\tilde{a} \triangleq (e^{\kappa(\theta_0 - t_0)} - 1) + ae^{\kappa(\theta_0 - t_0)}. \quad (39)$$

Let $(t_*, x_*) \in \mathcal{W}$. From the definition of set $\mathcal{W}$ it follows that there exists $\alpha \in A_t$, $\eta \in \cup_{\nu \in \epsilon_t} \alpha(\nu)$ and $\theta \in [t_*, \theta_0]$ such that

$$((\theta_*, \varphi(\theta, t_*, x_*, \eta)) \in M \cap N) \& ((\xi, \varphi(\xi, t_*, x_*, \eta)) \in N \forall \xi \in [t_*, \theta_*]). \quad (40)$$

Then, $d(\theta_*, \varphi(\theta_*, t_*, x_*, \eta_*)) \leq a$; see (37). As a corollary,

$$\|\varphi(\theta_*, t_*, x_*, \eta_*)\| \leq a.
By (34), (39), and (40) we obtain that $\|x_*\| \leq \tilde{a}$. Since the choice of $(t_*, x_*)$ was arbitrary, we have

$$\|x\| \leq \tilde{a} \forall (t, x) \in \mathcal{W}.$$ 

As a corollary, for $\tilde{a} \triangleq \sup\{\tilde{a}, \vartheta_0 - t_0\}$, we obtain that

$$d(t, x) \leq \tilde{a} \forall (t, x) \in \mathcal{W}.$$ 

From (37) and (38) we have that

$$d((t_1, x_1), (t_2, x_2)) \leq 2\tilde{a} \forall (t_1, x_1) \in \mathcal{W} \forall (t_2, x_2) \in \mathcal{W}.$$ 

4. The programmed iteration method. In this section, we consider some variants of programmed iterations for the $\mathcal{W}$ construction. Let us introduce a operator

$$A : \mathcal{P}(I_0 \times \mathbb{R}^n) \to \mathcal{P}(I_0 \times \mathbb{R}^n),$$

$$A(E) \triangleq \{(t, x) \in E \mid \forall \nu \in \mathcal{E}, \exists \eta \in \Pi_t[\nu] \exists \theta \in [t, \vartheta_0] : ((\theta, \varphi(\theta, t, x, \eta)) \in M) \& ((\xi, \varphi(\xi, t, x, \eta)) \in E \forall \xi \in [t, \theta]) \} \forall E \in \mathcal{P}(I_0 \times \mathbb{R}^n).$$

(41)

This operator corresponds to [24] of absorption set construction for the considered case.

In [29] the case of nonclosed (in our sense) set $N$ is considered: only the closedness of sections of $N$ is assumed. Therefore in [29] slightly different operator is used. Namely, it is used operator

$$\bar{A} : \mathcal{P}(I_0 \times \mathbb{R}^n) \to \mathcal{P}(I_0 \times \mathbb{R}^n);$$

the operator $\bar{A}$ is defined by the rule

$$\bar{A}(E) \triangleq \{(t, x) \in E \mid \forall \nu \in \mathcal{E}, \exists \eta \in \Pi_t[\nu] \exists \theta \in [t, \vartheta_0] : ((\theta, \varphi(\theta, t, x, \eta)) \in M) \& ((\xi, \varphi(\xi, t, x, \eta)) \in E \forall \xi \in [t, \theta]) \} \forall E \in \mathcal{P}(I_0 \times \mathbb{R}^n)$$

(42)

REMARK 4. If $E, E \subset I_0 \times \mathbb{R}^n$, is a closed set in $I_0 \times \mathbb{R}^n$, then

$$A(E) = \bar{A}(E).$$

(43)

Really, $A(E) \subset \bar{A}(E)$. Let $(t_*, x_*) \in \bar{A}(E)$. Then,

$$(t_*, x_*) \in E$$

(44)
and

$$\forall \nu \in \mathcal{E}_t \exists \eta \in \Pi_{t_0}[\nu] \exists \theta \in [t_*, \theta_0] :$$

$$((\theta, \varphi(\theta, t_*, x_*, \eta)) \in M) \& ((\xi, \varphi(\xi, t_*, x_*, \eta)) \in E \forall \xi \in [t_*, \theta]) \cap ((\xi, \varphi(\xi, t_*, x_*, \eta)) \in E \forall \xi \in [t_*, \theta_*]).$$

Fix $\nu_* \in \mathcal{E}_t$ and choose $\eta_* \in \Pi_{t_0}[\nu_*]$ and $\theta_* \in [t_*, \theta_0]$ for which

$$((\theta_*, \varphi(\theta_*, t_*, x_*, \eta_*)) \in M) \& ((\xi, \varphi(\xi, t_*, x_*, \eta_*)) \in E \forall \xi \in [t_*, \theta_*]). \quad (45)$$

If $t_* < \theta_*$, then from (45) we have $(\xi, \varphi(\xi, t_*, x_*, \eta_*)) \in E$ under all $\xi \in [t_*, \theta_*]$ since $E$ is a closed set and $\varphi(\cdot, t_*, x_*, \eta_*)$ is continuous function. If $\theta_* = t_*$, then $[t_*, \theta_*] = \{t_*\}$ and by (44)

$$(t_*, \varphi(t_*, t_*, x_*, \eta_*)) = (t_*, x_*) \in E;$$

so for any $\xi \in [t_*, \theta_*]$, $(\xi, \varphi(\xi, t_*, x_*, \eta_*)) \in E$ in the case $t_* = \theta_*$ too. So by (45) we obtain that

$$((\theta_*, \varphi(\theta_*, t_*, x_*, \eta_*)) \in M) \& ((\xi, \varphi(\xi, t_*, x_*, \eta_*)) \in E \forall \xi \in [t_*, \theta_*])$$

in all possible cases. Since the choice of $\nu_*$ is arbitrary we have the inclusion (see (41))

$$(t_*, x_*) \in A(E).$$

Therefore, $\tilde{A}(E) \subset A(E)$. The inverse inclusion is established above. Consequently, the equality (43) is established.

Let us remark the following representation: if $F$ is a closed subset of $N$ then

$$A(F) = \{(t, x) \in F| \pi_{t,x}(\nu|F) \neq \emptyset \forall \nu \in \mathcal{E}_t\}.$$  

Denote by $\mathcal{F}$ the family of all closed (in $I_0 \times \mathbb{R}^n$) subsets of $N$. Of course, $\mathcal{F}$ coincides with the family of all intersections $F \cap N$ where $F$ is a closed subset of $I_0 \times \mathbb{R}^n$. Then (see (43); [29, p. 25,57]) we have the following

**The invariance property:** $A(F) \in \mathcal{F} \forall F \in \mathcal{F}$.

For the sake of completeness we recall that (see 43)

$$A(F) = \tilde{A}(F) \forall F \in \mathcal{F}.$$  

Let the sequence

$$(W^{(k)})_{k \in \mathbb{N}_0} : \mathbb{N}_0 \to \mathcal{F}$$
be given by the rule \([29], [7]\):

\[
(W^{(0)} \triangleq N), \& (W^{(k)} = A(W^{k-1}) \forall k \in \mathcal{N}).
\]

Obviously,

\[
W^{(k+1)} \subset W^{(k)} \forall k \in \mathcal{N}_0.
\] (46)

Suppose

\[
W \triangleq \bigcap_{k \in \mathcal{N}_0} W^{(k)}. \tag{47}
\]

From (46) and (47) we have the usual set-theoretical monotone convergence

\[
(W^{(k)})_{k \in \mathcal{N}} \downarrow W. \tag{48}
\]

Due to the theorem 4.1 \([29]\) the following properties are valid:

1. \(W = \mathcal{W}\)
2. \(\forall L \in \mathcal{F} ((A(L) = L) \Rightarrow (L \subset W))\).

Let us consider another programmed iteration method called the stability iteration method (see \([29, \S 11]\)). If \(v \in Q\) then let \(A^{(v)}\) be a operator on the family \(\mathcal{P}(I_0 \times \mathbb{R}^n)\) such that

\[
A^{(v)}(E) \triangleq \{(t, x) \in E| \exists \mu \in \mathcal{R}_v \exists \theta \in [t, \theta_0] : ((\theta, \varphi_0(\theta, t, x, \mu)) \in M) \& ((\xi, \varphi_0(\xi, t, x, \mu)) \in E \forall \xi \in [t, \theta])\}. \tag{49}
\]

Of course, (49) defines the transformation of \(\mathcal{P}(I_0 \times \mathbb{R}^n)\). Following to \([29]\), we introduce the operator

\[
A : \mathcal{P}(I_0 \times \mathbb{R}^n) \rightarrow \mathcal{P}(I_0 \times \mathbb{R}^n) \tag{50}
\]

such that

\[
A(E) = \bigcap_{v \in Q} A^{(v)}(E). \tag{51}
\]

By the definition of operator \(A\) (50) (51) the iterated procedure of \([29, \S 11]\) is realized:

\[
(W_0 \triangleq N) \& (W_k = A(W_{k-1}) \forall k \in \mathcal{N}). \tag{52}
\]

For the sequence (52) we obtain that (see \([29, \S 11]\)) the convergence

\[
(W_k)_{k \in \mathcal{N}} \downarrow \mathcal{W}. \tag{53}
\]
In particular, $\mathcal{M}$ is the intersection of all sets $W_k$, $k \in \mathcal{N}_0$.

We note that, for any $v \in Q$ and $F \in \mathcal{F}$

$$A^{(v)}(F) = \{(t, x) \in F | \exists \nu \in E \exists \eta \in E_1 | \exists \theta \in [t, \theta_0] :$$

$$((\theta, \varphi(\theta, t, x, \eta)) \in M) \& ((\xi, \varphi(\xi, t, x, \eta)) \in F \forall \xi \in [t, \theta_0])\}.$$  \hspace{1cm} (54)

The proof of (54) is analogous to (43). Moreover, the following property is established in [36, §10]:

$$A(F) \in \mathcal{F} \forall F \in \mathcal{F}. \hspace{1cm} (55)$$

So, $\mathcal{F}$ is an invariant space of operators $A$ and $A$. Therefore, for any $k \in \mathcal{N}_0$

$$(W^{(k)} \in \mathcal{F}) \& (W_k \in \mathcal{F}). \hspace{1cm} (56)$$

Recall that

$$A(F) = \{(t, x) \in F | \forall \nu \in E_1 \exists \eta \in E_1 \exists \nu \exists \theta \in [t, \theta_0] :$$

$$((\theta, \varphi(\theta, t, x, \eta)) \in M) \& ((\xi, \varphi(\xi, t, x, \eta)) \in F \forall \xi \in [t, \theta])\} =$$

$$\{ (t, x) \in F | \forall \nu \in E_1 \exists \eta \in E_1 \exists \nu \exists \theta \in [t, \theta_0] :$$

$$((\theta, \varphi(\theta, t, x, \eta)) \in M) \& ((\xi, \varphi(\xi, t, x, \eta)) \in F \forall \xi \in [t, \theta])\} =$$

$$\{ (t, x) \in F | \forall \nu \in E_1 \exists \eta \in E_1 \exists \nu \exists \theta \in [t, \theta_0] :$$

$$((\theta, \varphi(\theta, t, x, \eta)) \in M \cap F) \& ((\xi, \varphi(\xi, t, x, \eta)) \in F \forall \xi \in [t, \theta])\} =$$

$$\{ (t, x) \in F | \forall \nu \in E_1 \exists \eta \in E_1 \exists \nu \exists \theta \in [t, \theta_0] :$$

$$((\theta, \varphi(\theta, t, x, \eta)) \in M \cap N) \& ((\xi, \varphi(\xi, t, x, \eta)) \in F \forall \xi \in [t, \theta])\} \forall F \in \mathcal{F}. \hspace{1cm} (57)$$

We have under $E \in \mathcal{P}(I_0 \times \mathbb{R}^n)$ (see [29, p. 60]) that

$$A^{(v)}(E) = \{(t, x) : \exists \eta \in E_1 | \lambda \in \mathcal{A}_1 | \exists \nu \exists \theta \in [t, \theta_0] :$$

$$((\theta, \varphi(\theta, t, x, \eta)) \in M) \& ((\xi, \varphi(\xi, t, x, \eta)) \in E \forall \xi \in [t, \theta_0]) \}. \hspace{1cm} (58)$$

By (51) and (58) we obtain that

$$A(E) = \{(t, x) \in E | \forall \nu \in Q \exists \eta \in E_1 | \lambda \in \mathcal{A}_1 | \exists \nu \exists \theta \in [t, \theta_0] :$$

$$((\theta, \varphi(\theta, t, x, \eta)) \in M) \& ((\xi, \varphi(\xi, t, x, \eta)) \in E \forall \xi \in [t, \theta]) \} \forall E \in \mathcal{P}(I_0 \times \mathbb{R}^n). \hspace{1cm} (59)$$

Then by (42), (43), and (59) the following property is fulfilled: if $F \in \mathcal{F}$, then

$$A(F) \subset A(F). \hspace{1cm} (60)$$
Therefore, from (46) and (52), (56), and (60) the following inclusion holds
\[ W^{(k)} \subset W_k \quad \forall k \in \mathcal{N}_0. \] (61)

Indeed, \( A \) and \( A \) is isotone operators. From (46) and (52)
\[ W^{(0)} = N = W_0. \]

If \( s \in \mathcal{N}_0 \), and \( W^{(s)} \subset W_s \) by (60) we obtain
\[ W^{(s+1)} = A(W^{(s)}) \subset A(W_s) = W_{s+1}. \]

By analogy with (57) from (59) one can obtain that for any \( F \in \mathcal{F} \)
\[ A(F) = \{(t, x) \in F | \forall v \in Q \exists \eta \in \Pi_t[\lambda_t \odot v] \exists \theta \in [t, \theta_0] : \]
\[ (((\theta, \varphi(\theta, t, x, \eta)) \in M) \& ((\xi, \varphi(\xi, t, x, \eta)) \in F \forall \xi \in [t, \theta])) = \]
\[ = \{(t, x) \in F | \forall v \in Q \exists \eta \in \Pi_t[\lambda_t \odot v] \exists \theta \in [t, \theta_0] : \]
\[ (((\theta, \varphi(\theta, t, x, \eta)) \in M \cap F) \& ((\xi, \varphi(\xi, t, x, \eta)) \in F \forall \xi \in [t, \theta])) = \]
\[ = \{(t, x) \in F | \forall v \in Q \exists \eta \in \Pi_t[\lambda_t \odot v] \exists \theta \in [t, \theta_0] : \]
\[ (((\theta, \varphi(\theta, t, x, \eta)) \in M \cap N) \& ((\xi, \varphi(\xi, t, x, \eta)) \in F \forall \xi \in [t, \theta])) \}. \] (62)

5. Hausdorff convergence. We use the compactness property (35) and (62). Recall that \( \tilde{a} = \sup\{\{a; \vartheta_0 - \iota_0\} \) where \( \tilde{a} \) is defined by (40); \( \tilde{a} \in [0, \infty[. \)

**Lemma 1.** \( \|x\| \leq \tilde{a} \forall (t, x) \in W_1. \)

**Proof.** By (62) we obtain that
\[ W_1 = A(N) = \{(t, x) \in N | \forall v \in Q \exists \eta \in \Pi_t[\lambda_t \odot v] \exists \theta \in [t, \theta_0] : \]
\[ (((\theta, \varphi(\theta, t, x, \eta)) \in M \cap N) \& ((\xi, \varphi(\xi, t, x, \eta)) \in N \forall \xi \in [t, \theta])) \}. \] (63)

Let \( (t_*, x_*) \in W_1 \). In particular, \( (t_*, x_*) \in I_0 \times \mathbb{R}^n \). Recall that \( Q \neq \emptyset \). Let us choose \( v_* \in Q \). By (63) there exist \( \eta_* \in \Pi_{t_*}[\lambda_{t_*} \odot v_*] \) and \( \theta_* \in [t_*, \theta_0] \) for which
\[ ((\theta_*, \varphi(\theta_*, t_*, x_*, \eta_*)) \in M \cap N). \] (64)

From (37) and (64) we obtain
\[ \|\varphi(\theta_*, t_*, x_*, \eta_*)\| \leq d(\theta_*, \varphi(\theta_*, t_*, x_*, \eta_*)) \leq a. \] (65)

By (34), (39), and (65) we have the inequality \( \|x_*\| \leq \tilde{a}. \) \( \square \)
CONVERGENCE OF PROGRAMMED ITERATION METHOD

COROLLARY 1. \( d(t, x) \leq \hat{a} \forall (t, x) \in W_1 \).

COROLLARY 2. If \( k \in \mathcal{N} \) and \( (t, x) \in W_k \), then \( (\|x\| \leq \hat{a}) \&(d(t, x) \leq \hat{a}) \).

The proof is obvious.

COROLLARY 3. For \( k \in \mathcal{N} \) and \( (t, x) \in W^{(k)} \),

\[
(\|x\| \leq \hat{a}) \&(d(t, x) \leq \hat{a}).
\]

The last corollary follows from (61) and Corollary 2.

PROPOSITION 1. For any \( k \in \mathcal{N} \) \( W_k \) is a compact set in \((I_0 \times \mathbb{R}^n, d)\).

Proof turns out by combination of (56) and corollary 2.

PROPOSITION 2. For any \( k \in \mathcal{N} \) \( W^{(k)} \) is a compact set in \((I_0 \times \mathbb{R}^n, d)\).

Proof follows from (56) and corollary 3.

PROPOSITION 3. \( \mathcal{W} \) is a compact set in \((I_0 \times \mathbb{R}^n, d)\). The proof follows from (54) and proposition 1.

As usual the neighborhood of the set \( \mathcal{W} \) is any open set containing \( \mathcal{W} \).

THEOREM 1. If \( G \) is a neighborhood of the set \( \mathcal{W} \) then there exists \( m \in \mathcal{N} \) such that \( W_k \subset G \) for all \( k \in \overline{m, \infty} \).

The proof follows from proposition 1 and the corollary 3.1.5 [28].

COROLLARY 4. If \( G \) is a neighborhood of the set \( \mathcal{W} \) then there exists \( m \in \mathcal{N} \) such that \( W^{(k)} \subset G \) for all \( k \in \overline{m, \infty} \).

The proof is obtained from (61) and previous theorem.

Denote by \( \rho \) the metric in \( \mathbb{R}^n \) generated by Euclidian norm \( \| \cdot \| \). For any \( F \in \mathcal{P}(I_0 \times \mathbb{R}^n) \) and \( t \in [t_0, \vartheta_0] \) define

\[
F[t] \triangleq \{ x \in \mathbb{R}^n | (t, x) \in F \}.
\] (66)

It is follows from previous statements that if \( t \in [t_0, \vartheta_0] \) then

1. \( W_k[t] \) is a compact set in \((\mathbb{R}^n, \rho)\) for any \( k \in \mathcal{N} \) and \( t \in [t, \vartheta_0] \);
2. \( W^{(k)}[t] \) is a compact set in \((\mathbb{R}^n, \rho)\) for any \( k \in \mathcal{N} \) and \( t \in [t, \vartheta_0] \);
3. \( \mathcal{W}[t] \) is a compact set in \((\mathbb{R}^n, \rho)\) for any \( t \in [t, \vartheta_0] \).

Let \( t \in [t_0, \vartheta_0] \). Usually, an open set \( H \subset \mathbb{R}^n \) is called a neighborhood of the section \( \mathcal{W}[t] \) if \( \mathcal{W}[t] \subset H \).

COROLLARY 5. If \( H \) is a neighborhood of the set \( \mathcal{W}[t] \) then there exists \( m \in \mathcal{N} \) such that \( W_k[t] \subset H \) for all \( k \in \overline{m, \infty} \).

This corollary follows from 2, 3, and (66).

COROLLARY 6. If \( H \) is a neighborhood of the set \( \mathcal{W}[t] \) then there exists \( m \in \mathcal{N} \) such that \( W^{(k)}[t] \subset H \) for all \( k \in \overline{m, \infty} \).

This corollary follows from theorem 4 and (66).

We assume up to the end of this section that

\[
M \cap \mathcal{N} \neq \emptyset.
\] (67)
Then (see (67) and Proposition 1) $\mathcal{W} \in \text{comp}([I_0 \times \mathbb{R}^n; d]$ and

$$W_k \in \text{comp}([I_0 \times \mathbb{R}^n; d] \forall k \in \mathcal{N}) \& W^{(k)} \in \text{comp}([I_0 \times \mathbb{R}^n; d] \forall k \in \mathcal{N}).$$

Consequently (see (1)), for any $k \in \mathcal{N}$, the values

$$(H_{d}(W_k, \mathcal{W})) \in [0, \infty] \& (H_{d}(W^{(k)}, \mathcal{W})) \in [0, \infty]$$

is well defined. And moreover the following statement is valid

THEOREM 2.

$$((H_{d}(W_k, \mathcal{W}))_{k \in \mathcal{N}} \rightarrow 0) \& ((H_{d}(W^{(k)}, \mathcal{W}))_{k \in \mathcal{N}} \rightarrow 0).$$

The proof is obvious (see Theorem 1 and corollary 1).

6. Approximative realization of stable bridge and Krasovskii-Subbotin extremal shift rule. In this section we construct step-by-step motions generating the positional strategies extremal to the defined above set $W^{(k)}$ and $W_k$. This rule is analog of Krasovskii-Subbotin extremal shift [1]. Considering sets are nonstable, but they are close to the set $\mathcal{W}$. This property guarantees the closeness of constructed motions to the set $M$ at some moment $t$.

Let $\Theta \triangleq \{t \in I_0 | \mathcal{W}[t] \neq \emptyset\}$.

PROPOSITION 4. If $\vartheta \in I_0$, then the following three properties are equivalent

1. $\vartheta \in \Theta$.
2. $W^{(k)}[\vartheta] \neq \emptyset \forall k \in N_0$
3. $W_k[\vartheta] \neq \emptyset \forall k \in N_0$

Proof. From (48) we obtain that $1. \Rightarrow 2$. From (61), the implication $2. \Rightarrow 3.$ follows. Let property $3.$ be valid. Recall that, from (54),

$$(W_k[\vartheta])_{k \in \mathcal{N}} \downarrow \mathcal{W}[\vartheta]. \tag{68}$$

In addition, the family $\{W_k[\vartheta] : k \in \mathcal{N}\}$ is a centered family of closed sets in the compact space $W_1[\vartheta]$ with the relative topology induced from $\mathbb{R}^n$ with usual topology of coordinate-wise convergence; in this connection recall that $W_1[\vartheta]$ is a compact set. Then by (68) we have $\mathcal{W}[\vartheta] \neq \emptyset$, so $\vartheta \in \Theta$. We obtain the implication $3. \Rightarrow 1$. □

We obtain that for $k \in \mathcal{N}$ and $\vartheta \in \Theta$ the values $H_{\rho}(W_k[\vartheta], \mathcal{W}[\vartheta])$, and $H_{\rho}(W^{(k)}[\vartheta], \mathcal{W}[\vartheta])$ is well defined.

THEOREM 3. If $\vartheta \in \Theta$, then

$$((H_{\rho}(W_k[\vartheta], \mathcal{W}[\vartheta]))_{k \in \mathcal{N}} \rightarrow 0) \&$$
The proof follows from the corollaries 5, 6.

Now let us turn to the Krasovskii-Subbotin shift extremal to the constructed sets $\{(W_k)_{k \in \mathbb{N}}\}$ and $\{(W^k)_{k \in \mathbb{N}}\}$. Further we suppose that

$$N = I_0 \times \mathbb{R}^\mathbb{N}.$$ 

Moreover we assume the following condition:

$$\min_{u \in \mathcal{P}} \max_{v \in \mathcal{Q}} v \in Q < s, f(t, x, u, v) \geq \max_{u \in \mathcal{P}} \min_{v \in \mathcal{Q}} v \in Q < s, f(t, x, u, v) \quad \forall s \in \mathbb{R}^n \forall (t, x) \in I_0 \times \mathbb{R}^n.$$ 

This condition is called Isaacs condition or saddle point condition in a small game.

Consider position $(t_*, x_*)$. Let $\Delta = \{\tau_j\}_{j=0}^N$ be a partition of the segment $[t_*, \theta_0]$. The control of player I is constant at the semiintervals $[\tau_{r-1}, \tau_r[$ and formed by the following rule. If $x_r$ is a position of system at the moment $\tau_r$, and $y^k_r$ is a closest to the point $x_r$ element of $W_k[\tau_r]$, then define $u^k_r$ by the rule

$$\max_{v \in \mathcal{Q}} v \in Q < x_r - y^k_r, f(\tau_r, x_k, u^k_r, v) \geq \min_{u \in \mathcal{P}} \max_{v \in \mathcal{Q}} v \in Q < x_r - y^k_r, f(\tau_r, x_r, u, v). \quad (69)$$

Now let us define step-by-step motions. This definition is almost idem to the definition given by N. N. Krasovskii and A. I. Subbotin [1, §1.2]. Let $v[\cdot]$ be a measurable control of a player II. A measurable function $x[\cdot]$ satisfied the following conditions

$$x[t_*] = x_*,$$

$$x_\Delta[t] = x_\Delta[\tau_r] + \int_{\tau_r}^{t} f(\theta, x_\Delta[\theta], u^k_r, v[\theta])d\theta, \quad t \in [\tau_r, \tau_{r+1}], \quad r = 0, N - 1$$

is called step-by-step motion. Denote the set of step-by-step motions constructed for $(t_*, x_*)$, $\Delta$ under the overrun of all possible controls $v[\cdot]$ by $X_{\Delta,k,x_*}$.

The following theorem is fulfilled.

**Theorem 4.** Let $\tau_* \in I_0$ be a moment such that $\mathcal{W}[\tau_*] \neq \emptyset$, and let $\varepsilon > 0$. Then there exist $\delta > 0$ such that for any partition $\Delta = \{\tau_r\}_{r=0}^N$ of segment $[\tau_*, \theta_0]$, satisfying the condition

$$\max_{r=0,N-1} (\tau_{r+1} - \tau_r) \leq \delta,$$
one can choose $J \in \mathcal{N}$ with property for all $j > J$ and $x_* \in W_j[\tau_*]$

$$\exists \theta \in [\tau_*, \emptyset_0] : (\rho - \min)[x[\theta], M[\theta]] \leq \varepsilon \forall x[.] \in X_{\Delta, j, x_*}.$$

Now let us consider strategies extremal to the sets $W^{(k)}$.

Consider a position $(t_*, x_*)$ and let partition $\Delta = \{\tau_j\}_{j=0}^N$ be a partition of segment $[t_*, \emptyset_0]$. Define the controls of player I at the semiintervals by the following rule. If $x_\tau$ is a position of system at the moment $\tau_\tau$, and $y^{k}_{\tau}$ is a closest to the point $x_\tau$ element of $W^{(k)}[\tau_\tau]$, then $u^k_\tau$ is an element of $P$ such that

$$\max_{v \in Q} < x_\tau - y^k_{\tau}, f(\tau_\tau, x_\tau, u^k_\tau, v) > = \min_{u \in P} \max_{v \in Q} < x_\tau - y^k_{\tau}, f(\tau_\tau, x_\tau, u, v) >.$$ 

(70)

Let us denote the set of step-by-step motions generated by the player I extremal controls $(u^j_{\tau})_{j=0}^{N-1}$ by $X_{\Delta, k, x_*}$.

**Theorem 5.** Let $\tau_* \in I_0$ be a moment such that $\mathcal{W}[\tau_*] \neq \emptyset$, and let $\varepsilon > 0$. Then there exist $\delta > 0$ such that for any partition $\Delta = \{\tau_r\}_{r=0}^N$ of segment $[\tau_*, \emptyset_0]$, satisfying the condition

$$\max_{r=0, N-1} (\tau_{r+1} - \tau_r) \leq \delta,$$

one can choose $J \in \mathcal{N}$ with property for all $j > J$ and $x_* \in W^{(j)}[\tau_*]$

$$\exists \theta \in [\tau_*, \emptyset_0] : (\rho - \min)[x[\theta], M] \leq \varepsilon \forall x[.] \in X_{\Delta, j, x_*}.$$

The proof of theorems 4 and 5 is analogous to the proof of Krasovskii-Subbotin alternative theorem [1, Chapter 2].

**REFERENCES**


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