ASYMPTOTIC EXPANSIONS FOR SOLUTIONS OF
ORDINARY DIFFERENTIAL EQUATIONS WITH A
MULTIPLE TURNING POINT

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Abstract. In the present article we study the asymptotic behavior of solutions of a linear ordinary differential equation of order \( n \) with a turning point of degree \( k \), where \( k \) is a natural number. The given equation is analyzed as a perturbation of a model equation. The given equation is represented as an integral equation, which is expressed via the solutions of the model equation. Using successive approximations we obtain the asymptotic expansions of solutions of the original equation.

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This paper is dedicated to the research of the asymptotic behavior of solutions of the equation

\[
 l(y) = y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = \lambda x^k q(x)y,
\]

which is considered on a closed interval \([a, b]\), \( a \leq 0 \leq b \) as \(|\lambda| \to \infty\). Here \( k \) and \( n \) are natural numbers, \( n > 2 \), \( p_j(x) \), \( j = 1, 2, \ldots, n \), \( q(x) \) are infinitely differentiable functions, \( q(x) \neq 0 \) for \( x \in [a, b] \), \( \lambda \) – complex parameter.

The point \( x = 0 \), \( 0 \in [a, b] \) is called the turning point of the equation (1) of the degree \( k \).

When \( n = 2 \) and \( k > -2 \), asymptotic solutions of the equation (1) are given in [2]. It is known [5], that if \( k = 0 \), i.e. with no turning point, then asymptotic solutions of our equation can be found in terms of the model equation \( y^{(n)} - \lambda y = 0 \). Similarly, when \( k = 1 \), using Theorem 7.2 [4, ch. IV],

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we can get the result in terms of the equation \( y^{(n)} - \lambda xy = 0 \). In this paper, following the scheme indicated in [3] and [5], we will begin our research from the following equation which is a model of the given equation (1).

We shall be concerned with a model differential equation of the form

\[
y^{(n)} = \lambda x^k y
\]

with a turning point \( x = 0 \) of degree \( k \). The change of variable \( x = \lambda^{-\frac{1}{n+k}} z \) transforms (2) into a Turrittin’s equation

\[
y^{(n)}(z) = z^k y(z),
\]

which we consider for any \( z \in \mathbb{C} \).

The equation (3) was researched from different points of view by various authors (see, for example, [4], [6], [7]). However, here we offer a different approach. We use the integral representation of solutions of (3) in the form of the multiple integral [1].

Let

\[
u_0(z) = \int_\gamma \cdots \int_\gamma \exp \left[ -\frac{1}{n + k} \sum_{j=1}^{k} \xi_j^{n+k} + z \prod_{j=1}^{k} \xi_j \right] \xi_2 \xi_3^2 \cdots \xi_k^{k-1} d\xi_1 \cdots d\xi_k,
\]

where \( \gamma \) is the contour in the complex plane, which consists of rays \( l_0 = [0, \infty) \) and \( l_1 = (+\infty, \exp[\frac{2\pi i}{n+k}], 0], \gamma = l_0 \cup l_1 \). Using integration by parts, we can assure that the function \( u_0 \) is the solution of the equation (3). The expression (4) allows us to get asymptotics of the the Turrittin’s functions for large as well as for small \( z \). Also, this expression is distinguished by its simplicity of the integrand.

Note, that the equation (3) possesses the property of invariance with respect to the transformation \( z \rightarrow \omega z' \), where \( \omega \) is the \( n + k \) -th root of unity. Therefore, the functions

\[
u_l(z) = u_0(z \exp[\frac{i2\pi l}{n + k}]), \quad l = 0, 1, 2, \ldots, n + k - 1,
\]

are also solutions of the equation (3).

The investigation of the integral (4) by the saddlepoint method [3] (see ch.V and Theorem 7.2, ch.IV) brings us to the following result.

Let \( \xi = (\xi_1, \xi_2, \ldots, \xi_k), \quad S(\xi, z) = -\frac{1}{n + k} \sum_{j=1}^{k} \xi_j^{n+k} + z \prod_{j=1}^{k} \xi_j. \)
The saddlepoints of the function $S$ are derived from the equation $S'_\xi = 0$. The solution of this equation is the point $\xi$:

$$\xi_j = z^{1/n}, \ j = 1, 2, ..., k.$$

Let's assume, that

$$\xi_{j,0} = |z|^{1/n} \exp\left(\frac{j\omega}{n}\right), \ \xi_{j,1} = |z|^{1/n} \exp\left[i\left(\frac{\psi}{n} + \frac{2\pi}{n(n+k)}\right)\right], \ \psi = \arg z.$$

**Theorem 1.** Let $k \geq 1$. The asymptotic of the solution $u_0(z)$ of the equation (3) is equal to

$1^0$. The contribution of the saddlepoint $\xi_{(0)} = (\xi_{1,0}, ..., \xi_{k,0})$ for

$$-\frac{\pi}{n + k} + \varepsilon \leq \psi \leq -\frac{3\pi n}{2(n + k)} - \varepsilon.$$

$2^0$. The sum of contributions of the saddlepoints $\xi_{(\alpha)} = (\xi_{1,\alpha_1}, \xi_{2,\alpha_2}, ..., \xi_{k,\alpha_k})$, where $(\alpha) = (\alpha_1, \alpha_2, ..., \alpha_k), \ \alpha_j = 0, 1; \ j = 1, 2, ..., k$;

$$|\alpha| = |\alpha_1 + \alpha_2 + ... + \alpha_k|, \ 1 \leq |\alpha| \leq k - 1, \ \text{for}$$

$$-\frac{\pi(2|\alpha| + 1)}{n + k} + \varepsilon \leq \psi \leq -\frac{\pi(2|\alpha| - 1)}{n + k} - \varepsilon.$$

$3^0$. The contribution of the saddlepoint $\xi_{(1)} = (\xi_{1,1}, \xi_{2,1}, ..., \xi_{k,1})$, $(1) = (1, ..., 1)$ for

$$-\frac{\pi}{2} - \frac{3\pi k}{n + k} + \varepsilon \leq \psi \leq -\frac{\pi(2k - 1)}{n + k} - \varepsilon.$$

$4^0$. The sum of contributions of the saddlepoints $\xi_{(\alpha)}$, $|\alpha| = 0, 1, ..., k$ in the rest of the sectors.

Here, $\varepsilon > 0$ can be chosen arbitrarily small, independent of $z$. Formulas for contributions $V_{\alpha}$ of the points $\xi_{(\alpha)}$ have the following form:

$$V_{\alpha}(z) = (-1)^{\alpha_1} \left(\frac{2\pi}{n + k - 1}\right)^{k/2} z^{k(2-n-k)/2n} \exp\left[\frac{n}{n + k} |z|^{n+k} \exp\left(\frac{i\psi(n+k)}{n} + \frac{2\pi |\alpha|}{n}\right)\right]$$

$$\times \prod_{j=1}^{k} |z|^{\frac{j-1}{n}} \exp\left(\frac{i(\psi + \frac{2\pi}{n+k} \alpha_j)(j-1)}{n}\right) \left(1 + \sum_{i=1}^{\infty} a_{\alpha,i} z^{\frac{-(n+k)}{n}}\right)$$

(6)
for any $\alpha$, $0 \leq |\alpha| \leq k$.

For $z^{n-k} \sqrt{2}\left(2-(n-k)\right)$ the positive branch of the root is chosen, when $z \in (0, \infty)$. The expansions (6) are differentiable any number of times.

The functions $u_0, u_1, \ldots, u_{n-1}$ form the fundamental system of solutions for the equation (3). Their linear independence follows, for example, from the fact that they have different asymptotics for real $z \to +\infty$. The coefficients $a_{\alpha,j}$ are found from the recursion formulas, which we can get, if we insert asymptotic series (6) into equation (3).

The same integral representation (4) allows us to find the asymptotic solutions of the equation (3) as $z \to 0$. In fact, differentiating the integral (4) with respect to $z$ $\nu$-times ($\nu \geq 1$) we will get the following expression at $z = 0$

$$
\left( \frac{\partial}{\partial z} \right)^\nu u_0(z) = \int_{\gamma} \cdots \int_{\gamma} \exp\left[-\frac{1}{n+k} \sum_{j=1}^{k} \xi_j^{n+k} \right] \prod_{j=1}^{k} \xi_j^{\nu} \xi_1^{\nu} \cdots \xi_k^{\nu} \, d\xi_1 \cdots d\xi_k
$$

$$
= \prod_{j=1}^{k} \int_{\gamma} \exp\left[-\frac{1}{n+k} \xi_j^{n+k} \right] \xi_j^{\nu} \, d\xi = \prod_{j=1}^{k} I_{j,\nu}.
$$

Integrals in $I_{j,\nu}$ along rays $l_0$ and $l_1$ are only distinguished by the factor

$$
\exp\left(2\pi i (\nu + j - 1)/(n+k) \right).
$$

But the integral along the ray $[0, \infty]$ equals

$$
\int_{0}^{\infty} \exp\left[-\frac{1}{n+k} \xi^{\nu+j-1} \right] \, d\xi = (n+k)^{\frac{\nu+j}{n+k}-1} \Gamma\left(\frac{\nu+j}{n+k}\right).
$$

Then,

$$
I_{j,\nu} = \left( \exp\left(2\pi i (\nu + j - 1)/(n+k) \right) - 1 \right) (n+k)^{\frac{\nu+j}{n+k}-1} \Gamma\left(\frac{\nu+j}{n+k}\right),
$$

and a product of $k$ of these terms, gives the expression for $\left( \frac{\partial}{\partial z} \right)^\nu u_0(z)$, i.e. we get the coefficient of $z^\nu$ in asymptotic expansions for $u_0(z)$ as $z \to 0$. Using formulas (5) we will obtain the asymptotic of the functions $u_1$ at $z = 0$.

Let’s go back to equation (1). No generality is lost in assuming that $p_1(x) \equiv 0, q(x) \equiv 1$. In fact, the substitution $x = \varphi(t), \varphi'(t) \neq 0$ changes the
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equation (1) into a equation with right side \( \lambda (\varphi'(t))^n \varphi^k(t) \times q(\varphi(t)) \times y(\varphi(t)) \). Because the function \( q(x) \neq 0 \) on \([a, b]\), we can always chose \( \varphi \) so that the expression \( (\varphi'(t))^n \varphi^k(t) \times q(\varphi(t)) = \text{sign}(q(0)) \times t^k \) would be true. Then the equation (1) transforms into

\[
(7) \quad \hat{y}^{(n)}(t) + \hat{p}_1(t)\hat{y}^{(n-1)}(t) + \ldots + \hat{p}_n(t)\hat{y}(t) = \lambda \times \text{sign}(q(0)) \times t^k \hat{y}(t).
\]

The transformation \( \hat{y}(t) = \exp\left(-\frac{1}{n} \int \hat{p}_1(t) dt\right) y^*(t) \) allows us to eliminate the term with \((n-1)\)-th derivative from (7). If we take \( \lambda' = \text{sign}(q(0)) \times \lambda \) as the new parameter, we eliminate the \( \text{sign}(q(0)) \) from the equation (7). Thus, we consider the equation

\[
(8) \quad y^{(n)} - \lambda x^k y = f(y),
\]

where \( f(y) = -(p_2 y^{(n-2)} + \ldots + p_n y) \). First, we will study this equation on the right interval \([0, b]\), then on the left interval \([a, 0]\). The fundamental system of solutions of the equation (8) is built using solutions of the model equation (2).

Let \( l \) be a ray, \( l : \arg \lambda = \varphi \). Let \( v_0(x, \lambda), v_1(x, \lambda), \ldots, v_j(x, \lambda), \ldots, v_{n-1}(x, \lambda) \) be solutions of the model equation (2). Let \( \omega_0, \omega_1, \ldots, \omega_{n-1} \) be the different \( n \)-th roots of unity. Then,

\[
(9) \quad v_j(x, \lambda) = u_0 \left( |\lambda|^{\frac{1}{n+k}} x \exp \left[ \frac{i \varphi + \arg \omega_j \times n}{n + k} \right] \right).
\]

Now, using Theorem 1, we have the following result about the asymptotic behavior of solutions of the equation (2).

**Theorem 2.** For every ray \( l \) of the complex plane, \( l : \arg \lambda = \varphi \), can be found a \( \theta(\varphi), |\theta(\varphi)| = 1 \), such that for solutions \( v_j(x, \lambda) \) we have the following asymptotic formulas

\[
v_j(x, \lambda) \sim \left( |\lambda|^{\frac{1}{n+k}} x \right)^{\frac{k(1-n)}{2n}} \exp \left[ \frac{n}{n + k} \left( |\lambda|^{\frac{1}{n+k}} x \right)^{\frac{n+k}{n}} \theta(\varphi) \omega_j \right] \times b_{j,0}(x, \lambda)
\]

(10)

as \( |\lambda|^{\frac{1}{n+k}} x \to \infty \), where \( b_{j,0}(x, \lambda) \) are infinitely differentiable functions with respect to \( x \) and \( \lambda \), and \( b_{j,0}(x, \lambda) \) together with their derivatives are bounded with respect to \( \lambda \). For the functions \( b_{j,0}(x, \lambda) \), we have the following asymptotic series

\[
b_{j,0}(x, \lambda) \sim \sum_{l=0}^{\infty} a_{j,l,0}(\varphi) \left( |\lambda|^{\frac{1}{n+k}} x \right)^{\frac{l(n+k)}{n}}
\]
as $x |\lambda|^{-\frac{1}{n+k}} \to \infty$. The asymptotic formulas (10) can be differentiated with respect to $x$ and $\lambda$ any number of times, when $\lambda \neq 0$.

We fix the direction of $l$ and chose the numbers $\omega_0$, $\omega_1$, ..., $\omega_{n-1}$, such that the following inequalities will be true

$$\text{Re}(\theta(\varphi)\omega_0) \leq \text{Re}(\theta(\varphi)\omega_1) \leq \ldots \leq \text{Re}(\theta(\varphi)\omega_{n-1}).$$

Applying the variation of parameters method to equation (8), we will get the integral representation of this equation

$$y(x, \lambda) = \sum_{j=0}^{n-1} c_j v_j(x, \lambda) + \frac{1}{2} \int_0^x \sum_{j=0}^{n-1} v_j(x, \lambda) \frac{W_j(x, \lambda)}{W(\lambda)} f(y(\xi, \lambda)) d\xi.$$

Here $W$ is the Wronskian of the functions $v_0$, $v_1$, ..., $v_{n-1}$:

$$W(\lambda) = \begin{vmatrix} v_0^{(n-1)} & \ldots & v_{n-1}^{(n-1)} \\ \vdots & \ddots & \vdots \\ v_0 & \ldots & v_{n-1} \end{vmatrix}.$$

The functions $W_0(x, \lambda)$, $W_1(x, \lambda)$, ..., $W_{n-1}(x, \lambda)$ are the cofactors of elements of the first row of the Wronskian (13). Since $p_1 \equiv 0$, the Wronskian (13) does not depend on $x$.

**Lemma 1.** The functions $W_j(x, \lambda)/W(\lambda)$ are infinitely differentiable with respect to $x$ and $\lambda$. For $\lambda \in \mathbb{I}$ we have the following inequalities

$$\left| \frac{d^s}{dx^s} \frac{W_j(x, \lambda)}{W(\lambda)} \right| \leq \frac{1}{\pi} \frac{n+k}{n-1} \frac{1}{n+k} (1 + |\lambda|^{-\frac{1}{n+k}} x)^{\frac{k(1-n)}{2n} + \frac{k-s}{n}} \times \left| \exp \left( -\frac{n}{n+k} \left( |\lambda|^{\frac{1}{n+k}} x \right)^{\frac{n+k}{n}} \theta(\varphi) \omega_j \right) \right|.$$

For the functions $W_j(x, \lambda)/W(\lambda)$, the following asymptotic series are valid:

$$W_j(x, \lambda)/W(\lambda) \sim \frac{|\lambda|^{-\frac{n-1}{n+k}} \left( |\lambda|^{\frac{1}{n+k}} x \right)^{\frac{k(1-n)}{2n}} \times \exp \left[ -\frac{n}{n+k} \left( |\lambda|^{\frac{1}{n+k}} x \right)^{\frac{n+k}{n}} \theta(\varphi) \omega_j \right]}{\sum_{l=0}^{\infty} W_{j,l,0}(\varphi) \times \left( x |\lambda|^{\frac{1}{n+k}} \right)^{\frac{i(n+k)}{n}}}$$
as $x |\lambda|^{\frac{1}{n+k}} \to \infty$. These expansions are differentiable any number of times.

Proof. By differentiating the asymptotic expansion (10) with respect to $x$, we will get

$\frac{d^s}{dx^s} v_j (x, \lambda) \sim \left( |\lambda|^{\frac{s}{n+k}} x \right)^{\frac{k}(2n) + \frac{s+1}{n}} |\lambda|^{\frac{s}{n+k}}$

\begin{align*}
(16) \quad & \times \exp \left[ \frac{n}{n+k} \left( |\lambda|^{\frac{1}{n+k}} x \right)^{\frac{n+k}{n}} \theta(\varphi) \omega_j \right] \times b_{j,s} (x, \lambda),
\end{align*}

where $b_{j,s} (x, \lambda)$ is an infinitely differentiable function with respect to $x$ and $\lambda$. It and all its derivatives are bounded with respect to $\lambda$, and it admits an asymptotic expansion

\begin{align*}
(17) \quad & b_{j,s} (x, \lambda) \sim \sum_{l=0}^{\infty} a_{j,l,s}\left( \varphi \right) \left( x |\lambda|^{\frac{1}{n+k}} \right)^{-\frac{k(n+k)}{n}}
\end{align*}

as $x |\lambda|^{\frac{1}{n+k}} \to \infty$. Then, according to (13), (16), and (17), we have

\begin{align*}
(18) \quad W(\lambda) \sim |\lambda|^{\frac{(n+1)n}{2(n+k)}} \left| \begin{array}{cccc}
 b_{0,n-1} & b_{1,n-1} & \cdots & b_{n-1,n-1} \\
 & & & \\
 & & & \\
 b_{0,0} & \cdots & b_{n-1,0}
\end{array} \right|
\end{align*}

as $x |\lambda|^{\frac{1}{n+k}} \to \infty$. The Wronskian is independent of $x$ and is not equal to zero, therefore the determinant in (18) also does not equal to zero. Next, we have

\begin{align*}
W_j (x, \lambda) \sim \prod_{s=0}^{n-2} \left[ |\lambda|^{\frac{s}{n+k}} \left( |\lambda|^{\frac{1}{n+k}} x \right)^{\frac{k(n)}{2n} + \frac{s+1}{n}} \right] \\
\times \prod_{t \neq j} \exp \left[ \frac{n}{n+k} \left( |\lambda|^{\frac{1}{n+k}} x \right)^{\frac{n+k}{n}} \theta(\varphi) \omega_t \right] \\
\times \left| \begin{array}{cccc}
 b_{0,n-2} & \cdots & b_{j-1,n-2} & b_{j+1,n-2} & \cdots & b_{n-1,n-2} \\
 & & & & & \\
 & & & & & \\
 b_{0,0} & \cdots & b_{j-1,0} & b_{j+1,n-2} & \cdots & b_{n-1,0}
\end{array} \right|
\end{align*}

(19)

as $x |\lambda|^{\frac{1}{n+k}} \to \infty$. Hence,

$$|W_j (x, \lambda)| \leq \left| C |\lambda|^{\frac{(n-2)(n-1)}{n+k}} \left( 1 + |\lambda|^{\frac{1}{n+k}} x \right)^{\frac{k(n-1)}{2n}} \right| \times \exp \left( -\frac{n}{n+k} \left( |\lambda|^{\frac{1}{n+k}} x \right)^{\frac{n+k}{n}} \theta(\varphi) \omega_j \right).$$
Using the inequality (20) and the asymptotic formula (18), we obtain the inequalities (14). The asymptotic formula (19) and expansion (17) bring us to asymptotic formula (15). □

Let \( m \) be a fixed number such that \( 0 \leq m \leq n - 1 \), and assume that

\[
c_j' = c_j, \text{ when } j = 0, 1, \ldots, m;
\]

(21) \[
c_j' = c_j + \frac{1}{2} \int_0^\infty \frac{W_j(x, \lambda)}{W(\lambda)} f(y(\xi, \lambda)) d\xi, \text{ when } j = m + 1, \ldots, n - 1.
\]

Then, the equation (12) takes the following form

\[
y(x, \lambda) = \sum_{j=0}^{n-1} c_j' v_j
\]

(22) \[+ \int_0^x K_1(x, \xi, \lambda) f(y(\xi, \lambda)) d\xi - \int_0^b K_2(x, \xi, \lambda) f(y(\xi, \lambda)) d\xi,
\]

where

(23) \[
K_1(x, \xi, \lambda) = \sum_{j=0}^m v_j(x, \lambda) \frac{W_j(x, \lambda)}{2W(\lambda)},
\]

(24) \[
K_2(x, \xi, \lambda) = \sum_{j=m+1}^{n-1} v_j(x, \lambda) \frac{W_j(x, \lambda)}{2W(\lambda)}.
\]

**Lemma 2.** The functions \( K_1, K_2 \) are infinitely differentiable, and for \( \lambda \in I \) the following bounds are valid

\[
\left| \frac{\partial^{a+b}}{\partial x^a \partial \xi^b} K_1(x, \xi, \lambda) \right| \leq C_{a,b} |\lambda|^{\frac{a+b+1-n}{n+k}} (1 + |\lambda|^{\frac{1}{n+k}} x)^{\frac{k(1-n)}{2n} + \frac{b\alpha}{n}}
\]

\[
\times \left( 1 + |\lambda|^{\frac{1}{n+k}} \xi \right)^{\frac{b\alpha}{n}} \left| \exp \left[ \frac{n}{n+k} |\lambda|^{\frac{1}{n+k}} \theta(\varphi) \omega_m \left( x^{\frac{n+k}{n}} - \xi^{\frac{n+k}{n}} \right) \right] \right|
\]

(25) for \( 0 \leq \xi \leq x \leq b; \)

\[
\left| \frac{\partial^{a+b}}{\partial x^a \partial \xi^b} K_2(x, \xi, \lambda) \right| \leq C_{a,b} |\lambda|^{\frac{a+b+1-n}{n+k}} (1 + |\lambda|^{\frac{1}{n+k}} x)^{\frac{k(1-n)}{2n} + \frac{b\alpha}{n}}
\]

\[
\times \left( 1 + |\lambda|^{\frac{1}{n+k}} \xi \right)^{\frac{b\alpha}{n}} \left| \exp \left[ \frac{n}{n+k} |\lambda|^{\frac{1}{n+k}} \theta(\varphi) \omega_m \left( x^{\frac{n+k}{n}} - \xi^{\frac{n+k}{n}} \right) \right] \right|
\]

(26)
for $0 \leq x \leq \xi \leq b$.

Also, for functions $K_1, K_2$ the following asymptotic expansions are true:

$$K_1(x, \xi, \lambda) \sim |\lambda|^{\frac{k(1-n)}{2n}} \left( x^\frac{1}{n+k} \right)^{\frac{k(1-n)}{2n}} \left( \xi^\frac{1}{n+k} \right)^{\frac{k(1-n)}{2n}} \sum_{j=0}^{m} \exp \left[ \frac{n}{n+k} |\lambda|^\frac{1}{n} \theta(\varphi) \omega_j \left( x^\frac{n+k}{n} - \xi^\frac{n+k}{n} \right) \right]$$

$$\times \sum_{l,s} k_{1,l,s}(\varphi) \left( x^\frac{1}{n+k} \right)^{-\frac{(l+n+k)}{n}} \left( \xi^\frac{1}{n+k} \right)^{-\frac{(l+n+k)}{n}} ;$$

$$K_2(x, \xi, \lambda) \sim |\lambda|^{\frac{k(1-n)}{2n}} \left( x^\frac{1}{n+k} \right)^{\frac{k(1-n)}{2n}} \left( \xi^\frac{1}{n+k} \right)^{\frac{k(1-n)}{2n}} \sum_{j=m+1}^{n} \exp \left[ \frac{n}{n+k} |\lambda|^\frac{1}{n} \theta(\varphi) \omega_j \left( x^\frac{n+k}{n} - \xi^\frac{n+k}{n} \right) \right]$$

$$\times \sum_{l,s} k_{2,l,s}(\varphi) \left( x^\frac{1}{n+k} \right)^{-\frac{(l+n+k)}{n}} \left( \xi^\frac{1}{n+k} \right)^{-\frac{(l+n+k)}{n}} .$$

The indicated asymptotic expansions are differentiable any number of times.

Proof. For the proof of (25) and (26) we use Lemma 1, representation (10), and also, inequalities

$$Re(\theta(\varphi)\omega_0) \leq Re(\theta(\varphi)\omega_1) \leq \ldots \leq Re(\theta(\varphi)\omega_m)$$

and

$$Re(\theta(\varphi)\omega_m) \leq Re(\theta(\varphi)\omega_{m+1}) \leq \ldots \leq Re(\theta(\varphi)\omega_{n-1})$$

respectively.

The lemma is proved. □

Let $l$: $\arg \lambda = \varphi$. Then, using Theorem 1, we can find for the function

$$v_m(x, \lambda) = u_0 \left( |\lambda|^{\frac{1}{n+k}} x \exp \left[ \frac{i \varphi + \arg \omega_m \times n}{n + k} \right] \right)$$

a multi-index $\alpha_m$, such that the saddlepoint $\xi_{\alpha_m}$ determines the main contribution $V_{\alpha_m}$ to the asymptotic representation of the function $v_m(x, \lambda)$. In general, there are finite number of rays, on which this contribution is defined by several saddlepoints. In this case for $\alpha_m$ we chose an arbitrary multi-index, which determines one of these saddlepoints.

Now, for the equation (1) we have the following result.

Theorem 3. The equation (1) has $n$ linearly independent solutions, for which on each ray $l$ of the complex plane, $l$: $\arg \lambda = \varphi$, the following asymptotic expansions are valid

$$(27) \quad y_m(x, \lambda) \sim v_m(x, \lambda) + \sum_{j=1}^{\infty} g_{m,j}(x, \lambda) \lambda^{-j/n}$$
as \( x |\lambda|^{\frac{1}{n+k}} \to \infty \), where \( m = 0, 1, \ldots, n - 1 \), the functions \( g_{m,j}(x, \lambda) \) are infinitely differentiable with respect to \( x \) and \( \lambda \), and

\[
\frac{d^s}{dx^s} g_{m,j}(x, \lambda) = O \left( \frac{\exp \left( \frac{\lambda^{\frac{1}{n+k}} x}{n^{\frac{1}{n+k}} } \right)}{n - k} \right).
\]

The expansions (27) are differentiable any number of times with respect to \( x \) and \( \lambda \).

**Proof.** Let’s assume that the equation (8) has a solution \( y_m \) in the form (22) such, that \( c'_j = 0 \) for \( j \neq m \) and \( c'_m = 1 \). Then, it is a solution of the following integral equation

\[
y_m = v_m + \int_0^x K_1(x, \xi, \lambda) f(y_m(\xi, \lambda)) d\xi - \int_x^b K_2(x, \xi, \lambda) f(y_m(\xi, \lambda)) d\xi.
\]

(28)

Let \( y_m = v_m + r_m \). For defining the function \( r_m \) we consider the equation

\[
r_m = \int_0^x K_1(x, \xi, \lambda) f(v_m(\xi, \lambda)) d\xi - \int_x^b K_2(x, \xi, \lambda) f(v_m(\xi, \lambda)) d\xi + \int_0^x K_1(x, \xi, \lambda) f(r_m(\xi, \lambda)) d\xi - \int_x^b K_2(x, \xi, \lambda) f(r_m(\xi, \lambda)) d\xi.
\]

(29)

We solve equation (29) by the method of successive approximations. According to Theorem 2, using the infinite differentiability of the function \( v_m(\xi, \lambda) \) we obtain the bounds

\[
\left| \frac{d^s}{dx^s} v_m(x, \lambda) \right| \leq C_s |\lambda|^{\frac{1}{n+k}} \left( 1 + |\lambda|^{\frac{n}{n+k}} \right)^{\frac{k(1-n) + k\theta}{2n}} \exp \left( -\frac{\lambda^{\frac{1}{n+k}} x}{n^{\frac{1}{n+k}} } \right) \left| \theta(\varphi) \omega_m \right|
\]

(30)

Then, according to Lemma 2 and the expression (16) we get the inequality

\[
|\int_0^x K_1(x, \xi, \lambda) f(v_m(\xi, \lambda)) d\xi| \leq |\int_0^x K_1(x, \xi, \lambda) (-p_2(\xi) v_m^{n-2}(\xi, \lambda) - p_n(\xi) v_m(\xi, \lambda)) d\xi|
\]

\[
\leq C_{0,1} (1 + |\lambda|^{\frac{1}{n+k}} x)^{\frac{k(1-n)}{2n}} \times \exp \left( -\frac{\lambda^{\frac{1}{n+k}} x}{n^{\frac{1}{n+k}} } \right) \left| \theta(\varphi) \omega_m \right| |\lambda|^{-\frac{1}{n}}.
\]
Then, the corresponding bounds for the derivatives are
\[
\left| \frac{d^s}{dx^s} \int_0^\infty K_1(x, \xi, \lambda) f(v_m(\xi, \lambda))d\xi \right| \leq C_{s,1} |\lambda|^{-\frac{1}{n}} \left( 1 + |\lambda|^{\frac{1}{n+k}} x \right)^{\frac{k(1-n)}{2n} + \frac{s}{n}} |\lambda|^{\frac{n}{n+k}} \\
\times \left| \exp \left[ \frac{n}{n+k} \left( |\lambda|^{\frac{1}{n+k}} x \right)^{\frac{n+k}{n}} \theta(\varphi) \omega_m \right] \right|. 
\]
(31)

Similarly, we get the bounds for the second integral in the right side of the equation (28):
\[
\left| \frac{d^s}{dx^s} \int_x^b K_2(x, \xi, \lambda) f(v_m(\xi, \lambda))d\xi \right| \leq C_{s,1}^\prime |\lambda|^{-\frac{1}{n}} \left( 1 + |\lambda|^{\frac{1}{n+k}} x \right)^{\frac{k(1-n)}{2n} + \frac{s}{n}} |\lambda|^{\frac{n}{n+k}} \\
\times \left| \exp \left[ \frac{n}{n+k} \left( |\lambda|^{\frac{1}{n+k}} x \right)^{\frac{n+k}{n}} \theta(\varphi) \omega_m \right] \right|. 
\]
(32)

Let
\[
\tau_{m,0} = \int_0^\infty K_1(x, \xi, \lambda) f(v_m(\xi, \lambda))d\xi - \int_x^b K_2(x, \xi, \lambda) f(v_m(\xi, \lambda))d\xi. 
\]

With the help of the recursion relations we define
\[
\tau_{m,j} = \int_0^\infty K_1(x, \xi, \lambda) f(r_{m,j-1}(\xi, \lambda))d\xi - \int_x^b K_2(x, \xi, \lambda) f(r_{m,j-1}(\xi, \lambda))d\xi. 
\]
(33)

The inequalities (31) and (32), giving bounds for derivatives of the function \( \tau_{m,0} \), differ from bounds (30) for derivatives of the function \( v_m \) only by a factor \( |\lambda|^{-1/n} \). Therefore, for the estimation of \( \tau_{m,j} \) we use the same scheme, that we used for the estimation of \( \tau_{m,0} \). Thus, we get the bound
\[
\left| \frac{d^s}{dx^s} \tau_{m,j}(x, \lambda) \right| \leq C_{s,j+1} |\lambda|^{-\frac{j+1}{n}} \left( 1 + |\lambda|^{\frac{n}{n+k}} x \right)^{\frac{k(1-n)}{2n} + \frac{s}{n}} |\lambda|^{\frac{n}{n+k}} \\
\times \left| \exp \left[ \frac{n}{n+k} \left( |\lambda|^{\frac{1}{n+k}} x \right)^{\frac{n+k}{n}} \theta(\varphi) \omega_m \right] \right|, 
\]

where \( C_{s,j+1} = C_{s,j+1}^\prime + C_{s,j+1}'' \).

Therefore, using the method of successive approximations, we obtain a collection of the functions \( \tau_{m,j}(x, \lambda) \), the summation of which brings us to an asymptotic series in powers of \( |\lambda|^{-1/n} \). This series gives the solution of the equation (29).
All we have left to proof is that there exists a solution \( y_m \) of the equation (8), that satisfies the equation (28). Just like in [5, pp.60-61], it is sufficient to show, that for any constants \( c_j \) there exists a solution \( y \) of the equation (8) that satisfies (22) for these values of \( c_j \).

The equations (21) are the linear transformation from \( c_j \) to \( c_j' \); therefore, it is sufficient to prove that the determinant of this transformation is not equal to zero. We assume the contrary, then the equations (21) have nontrivial solutions with respect to \( c_j \), when \( c_0 = c_1' = \ldots = c_{n-1}' = 0 \). Then, the corresponding function \( y \) will be a nontrivial solution of the equation (22), i.e.

\[
y(x, \lambda) = \int_0^x K_1(x, \xi, \lambda)f(y(\xi, \lambda))d\xi - \int_0^b K_2(x, \xi, \lambda)f(y(\xi, \lambda))d\xi.
\]

We introduce the functions \( z_s(x, \lambda) \), \( s = 0, 1, \ldots, n - 1 \), such that

\[
\frac{d^s}{dx^s} y = |\lambda|^\frac{n}{n+k} \left( 1 + |\lambda|^\frac{1}{n+k} x \right) \frac{h(n-1)}{2n + \frac{nk}{n}}
\times \exp \left[ \frac{n}{n+k} \left( |\lambda|^\frac{1}{n+k} \right)^n \theta(\varphi) \omega_m \right] z_s.
\]

By differentiating (34) \( n - 1 \) times we get the following system of equations for \( z_s \)

\[
z_s = |\lambda|^\frac{n}{n+k} \left( 1 + |\lambda|^\frac{1}{n+k} x \right)^\frac{h(n-1)}{2n + \frac{nk}{n}}
\times \exp \left[ -\frac{n}{n+k} \left( |\lambda|^\frac{1}{n+k} \right)^n \theta(\varphi) \omega_m \right]
\times \left[ \frac{d^s}{dx^s} f_2 K_1(x, \xi, \lambda) \exp \left[ \frac{n}{n+k} \left( |\lambda|^\frac{1}{n+k} \xi \right)^n \theta(\varphi) \omega_m \right] \right]
\times \left[ -p_2(\xi) \lambda^\frac{n-1}{n+k} \left( 1 + |\lambda|^\frac{1}{n+k} \xi \right)^\frac{h(n-1)}{2n + \frac{nk}{n}} z_{n-2}(\xi, \lambda) \right] -
\]

\[
- p_n(\xi) \left( 1 + |\lambda|^\frac{1}{n+k} \xi \right)^\frac{h(n-1)}{2n + \frac{nk}{n}} z_0(\xi, \lambda) d\xi
\]

\[
- \frac{d^s}{dx^s} f_1 K_2(x, \xi, \lambda) \exp \left[ \frac{n}{n+k} \left( |\lambda|^\frac{1}{n+k} \xi \right)^n \theta(\varphi) \omega_m \right] \times
\]
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Let

\[ M(\lambda) = \max_{0 \leq x \leq b} |z_s(x, \lambda)|, \; s = 0, 1, \ldots, n - 2. \]

Then, similar to (31), we have the following bound for the expression (35)

\[ |z_s(x, \lambda)| \leq CM(\lambda) |\lambda|^{-1/n}. \]

Since, the maximum of the left side of this inequality is \( M \), then \( M(\lambda) \leq CM(\lambda) |\lambda|^{-1/n} \). This contradiction proves our theorem. \( \square \)

Thus, we constructed the fundamental system of solutions of the equation (1) and obtained the asymptotic expansions of these solutions as \( x |\lambda|^{-1/(n+k)} \to \infty \). However, for \( x = 0 \), we can use the formula (12) and the solutions from the constructed fundamental system will have the same values as the Turrittin’s functions. By differentiating (12) with respect to \( x \), at \( x = 0 \) we get an expression for the derivatives of these solutions through the derivatives of the Turrittin’s functions and the coefficients of the equation. Hence, we have found all the Cauchy’s data of the indicated solutions at \( x = 0 \).

Then, we can examine the equation (1) on the left interval \([a, 0], \; a \leq 0\) by the same method and construct on the indicated interval the asymptotic expansions of solutions of the fundamental system as \( x |\lambda|^{-1/(n+k)} \to \infty \), similar to (27). These solutions at \( x = 0 \) will have the same values as the the Turrittin’s functions. Here, the set of the the Turrittin’s functions could be taken the same as on the interval \([0, b]\); however, they could be placed in a different order. The values of the derivatives of these solutions at \( x = 0 \) are also explicitly expressed through derivatives of the the Turrittin’s functions and the coefficients of our equation.

Thus, because the “right” and the “left” expansions obtained in Theorem 3 are uniformly convergent and have the same Cauchy’s data at \( x = 0 \), they represent the same fundamental system of solutions. Hence, Theorem 3 covers an equation of the form (1), where the turning point is inside of the considered interval.
REFERENCES


