ON NON-CLASSICAL CAUCHY PROBLEM
FOR PARABOLIC FUNCTIONAL-DIFFERENTIAL
EQUATIONS WITH BESSEL OPERATORS *

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Abstract. This paper continues the investigations originated in [9]–[11]. We consider
the problem, stated in the title above, for the most general type of $B$-parabolic equations
i. e. for the equations with arbitrary numbers of special and non-special spatial variables.
We prove the unique solvability of the investigating problem, find the integral representa­
tion of its solution, and prove the theorems on (weighted asymptotic) closeness of the
investigating solution and the solution of a well-investigated problem.

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stabilization, singularities

1. Introduction. Theory of differential-difference equations containing
translation operators (apart from differential ones) is a rapidly developing
area of research (see e. g. [2], [4], [14], [16], [18], [19] and references therein).
Specified equations frequently arise in applications, for instance, in models of
non-linear optics (see [15], [18], [20]). On the other hand, theory of singular
differential equations containing Bessel operator is also quite actual nowadays
(see [5] and references therein). Those equations also arise in applications, in
particular, in non-isotropic models of mathematical physics where features
of the medium degenerate as powers.

In this work we investigate non-classical Cauchy problem for parabolic
functional-differential equation, where second derivatives and translation op­

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operators act with respect to other spatial variables (so-called-special variables). Let us note, that besides purely applied interest to generalize models of [15], [18], [20] to the case of medium with features, degenerating along selected directions, the studied problem is principally different from the classical setting in the theoretical aspect too since the equation is not only differential-difference now - it also becomes integrodifferential.

We will find fundamental solution of the specified equation, investigate its properties, and find (under the assumption of continuity and boundedness of the initial-value function and right-hand side) integral representation of solution of the considered problem. This will prove the existence theorem.

To prove the uniqueness theorem we use Fourier transforms method. Note that function-theory technique needed for that (Fourier-Bessel transformation and the scale of generalized functions associated with the degenerative measure \( \prod_{i} y_i^{k_i} dx dy \)) is broadly and deeply developed in [5] (see also references therein) so, according to the general scheme of [3], the specified method is also applicable to the investigation of solvability. However solutions, existence of which is proved by that method, are distributions. Moreover, in general they even do not belong to Schwartz space. In this paper we obtain classical solution i.e. function possessing classical derivatives of all necessary orders, which satisfies the equation and boundary-value condition at each point.

Further we study the long-time behaviour of the found solution and establish the theorems on the asymptotic closeness of solutions (in particular cases, stabilization theorems), taking place (as it is well-known) in the classical parabolic case either. However, here we found principally new effects, which do not occur in the classical case (see e.g. [1] and references therein), in the case of singular differential equation (see e.g. [12]), or in the case of differential-difference equation [9].

2. Notation and main definitions. We will use the following notation:

- \( k_l = 2\nu_l + 1 \) is a positive parameter \((l \in 1, n)\);
- \( B_{k_l,y_l} \equiv \frac{1}{y_l^{k_l}} \frac{\partial}{\partial y_l} (y_l^{k_l} \frac{\partial}{\partial y_l}) = \frac{\partial^2}{\partial y_l^2} + \frac{k_l}{y_l} \frac{\partial}{\partial y_l} \) is Bessel operator with respect to the variable \( y_l \);
- \( T^h_y f(y) \equiv \frac{\Gamma(\nu + 1)}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_{0}^{\pi} \left( \sqrt{y^2 + h^2 - 2yh \cos \theta} \right) \sin^{2\nu} \theta d\theta \) is the corresponding operator of generalized translation (here variable \( y \) is scalar);
- in case where \( y, h \) are vectors, generalized translation operator is defined as superposition of one-dimensional operators: \( T^h_y = T^h_{y_1} \cdots T^h_{y_n} \);
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\[ j_\nu(y) \overset{\text{def}}{=} \frac{2^\nu \Gamma(\nu + 1)}{y^\nu} J_\nu(y) \] is the normalized (in the uniform norm) Bessel function of the first kind.

\( \mathbb{R}^{m+n}_+ \) denotes the set \( \{(x, y) \mid x \in \mathbb{R}^m, y_1 > 0, \ldots, y_n > 0\} \).

In \( \mathbb{R}^{m+n}_+ \times (0, \infty) \) we consider equation

\[
\frac{\partial u}{\partial t} - \sum_{i=1}^{m} \left[ \frac{\partial^2 u}{\partial x_i^2} + \sum_{s=1}^{m_i} a_{is} u(x + h_{is}, y, t) \right] - \sum_{i=1}^{n} \left( B_{kl,y_i} u + \sum_{r=1}^{m_l} b_{ir} T_{y_i}^{gr} u \right) = f(x, y, t),
\]

with boundary-value conditions

\[
\left. \frac{\partial u}{\partial y_l} \right|_{y_l = 0} = 0 \quad (l = 1, n), \quad t > 0;
\]

\[
\left. u \right|_{t=0} = u_0(x, y), \quad (x, y) \in \mathbb{R}^{m+n}_+.
\]

Here \( u_0, f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_m}, \frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_n} \) are continuous and bounded, \( f \) also satisfies condition (2), for any \( s \) vectors \( h_{is} \) are parallel to the \( i \)th co-ordinate axis of space \( \mathbb{R}^m \) \( (i \in 1, m) \), coefficients \( a_{is}, b_{ir}, g_{ir} \) are supposed to be real for all values of their indices.

Let us note, that solution of non-classical Cauchy problem (1)-(3) is defined merely for \( (x, y) \) from \( \mathbb{R}^{m+n}_+ \) while in order to apply generalized translation operator we need, in general, the function to be defined for negative values of variable \( y_l \) \( (l \in 1, n) \) either; in that case even with respect to any \( y_l \) continuation of the solution is used - this is possible by the virtue of evenness condition (2). In other words, problem (1)-(3) may be considered in the whole \( \mathbb{R}^{m+n}_+ \times (0, +\infty) \) with the replacement of condition (2) by the assumption of evenness of function \( u \) with respect to any variable \( y_l \). For differential parabolic equations with Bessel operator above problems are well-posed (see e.g. [6]-[8] and references therein).

3. Definition of fundamental solution. Let \( f(x, y, t) \equiv 0 \).

On \( \mathbb{R}^{m+n}_+ \times (0, \infty) \) we define function \( \mathcal{E}(x, y, t) \overset{\text{def}}{=} \mathcal{E}_1(x, y, t) \mathcal{E}_2(x, y, t) \), where

\[
\mathcal{E}_1(x, t) \overset{\text{def}}{=} \int_{\mathbb{R}^m} e^{-t(|\xi|^2 - \sum_{i=1}^{m} \sum_{s=1}^{m_i} a_{is} \cos h_{is} \cdot \xi)} \cos \left( x \cdot \xi + t \sum_{i=1}^{m} \sum_{s=1}^{m_i} a_{is} \sin h_{is} \cdot \xi \right) d\xi,
\]
For any \( t_0, T \in (0, +\infty) \) integrals (4) and (5) converge absolutely and uniformly with respect to \((x, y, t) \in \mathbb{R}^{m+n} \times [t_0, T]\) (note that \(|j_r(z)| \leq 1\) so \(E(x, y, t)\) is well-defined. Let us substitute (formally) \(E\) into equation (1).

\[
\frac{\partial E}{\partial t} = \frac{\partial E_1}{\partial t} E_2 + E_1 \frac{\partial E_2}{\partial t}, \quad \frac{\partial^2 E}{\partial x_i^2} = \frac{\partial^2 E_1}{\partial x_i^2} E_2 (i = 1, m),
\]

\[
B_{k_1,y_l} E = E_1 \frac{\partial^2 E_2}{\partial y_l^2} + \frac{k_1}{y_l} E_1 \frac{\partial E_2}{\partial y_l} = E_1 B_{k_1,y_l} E_2 (l = 1, n),
\]

\(E(x + h, y, t) = E_1(x + h, y, t)E_2\) for any \(h \in \mathbb{R}^m\), \(T_y^g E = E_1T_y^g E_2(x, y, t)\) for any \(g \in \mathbb{R}^n\). Thus

\[
\frac{\partial E}{\partial t} - \sum_{i=1}^m \left[ \frac{\partial^2 E}{\partial x_i^2} + \sum_{s=1}^{m_i} a_{is} E(x + h_{is}, y, t) \right] = \sum_{l=1}^n \left( B_{k_1,y_l} E + \sum_{r=1}^{n_l} b_{r,y_l} T_{y_l}^{g_r} E \right) =
\]

\[
= E_2 \left[ \frac{\partial E_1}{\partial t} - \Delta_x E_1 - \sum_{i=1}^m \sum_{s=1}^{m_i} a_{is} E_1(x + h_{is}, y, t) \right] +
\]

\[
+ E_1 \left[ \frac{\partial E_2}{\partial t} - \sum_{l=1}^n \left( B_{k_1,y_l} E_2 + \sum_{r=1}^{n_l} b_{r,y_l} T_{y_l}^{g_r} E_2 \right) \right].
\]

It is known from [9] that the first term of (6) vanishes; let us consider the second term.

\[
\frac{\partial E_2}{\partial t} = \int_0^\infty \int_0^\infty \left[ \sum_{l=1}^n \sum_{r=1}^{n_l} b_{r,y_l} j_{y_l}(g_r \eta_l) - |\eta|^2 \right] e^{\left[ \sum_{l=1}^n \sum_{r=1}^{n_l} b_{r,y_l} j_{y_l}(g_r \eta_l) - |\eta|^2 \right] t} \times
\]

\[
\prod_{l=1}^n \eta_l^{k_1} j_{y_l}(y_l \eta_l) d\eta_l =
\]

\[
= \int_0^\infty \int_0^\infty \left[ \sum_{l=1}^n \sum_{r=1}^{n_l} b_{r,y_l} T_{y_l}^{g_r} j_{y_l}(y_l \eta_l) \prod_{\kappa \neq l} j_{y_\kappa}(y_\kappa \eta_\kappa) e^{\left[ \sum_{l=1}^n \sum_{r=1}^{n_l} b_{r,y_l} j_{y_l}(g_r \eta_l) - |\eta|^2 \right] t} \times
\]

\[
\prod_{\kappa=1}^{n_l} \eta_l^{k_1} j_{y_l}(y_l \eta_l) d\eta_l =
\]
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\[ \times \prod_{l=1}^{n} \eta_{l}^{k_{l}} d\eta_{l} - \frac{\infty}{\eta_{l}^{k_{l}} \prod_{\kappa \neq l}^{n} \eta_{l}^{k_{\kappa}} e^{[\sum_{i=1}^{m} b_{i} j_{\nu_{i}}(g_{i_{0}} \eta_{l}) - |\eta_{l}|^{2}]^{t}} \prod_{l=1}^{n} j_{\nu_{l}}(y \eta_{l}) d\eta_{l} }{n \times \eta_{l}^{k_{l}} \prod_{\kappa \neq l}^{n} \eta_{l}^{k_{\kappa}} e^{[\sum_{i=1}^{m} b_{i} j_{\nu_{i}}(g_{i_{0}} \eta_{l}) - |\eta_{l}|^{2}]^{t}} \prod_{l=1}^{n} j_{\nu_{l}}(y \eta_{l}) d\eta_{l} }\]

since \( T_{2}^{y_{l}} j_{\nu}(ax) = j_{\nu}(ax) j_{\nu}(ay) \) (see e.g. [5], p. 19).

Further, \( B_{k_{i},y_{l}} j_{\nu_{l}}(y \eta_{l}) = -\eta_{l}^{2} j_{\nu_{l}}(y \eta_{l}) \) for any \( l \) (see e.g. [5], p. 18) so

\[ B_{k_{i},y_{l}} \mathcal{E}_{2} = -\frac{\infty}{\prod_{\kappa \neq l}^{n} \eta_{l}^{k_{\kappa}} e^{[\sum_{i=1}^{m} b_{i} j_{\nu_{i}}(g_{i_{0}} \eta_{l}) - |\eta_{l}|^{2}]^{t}} \prod_{l=1}^{n} j_{\nu_{l}}(y \eta_{l}) d\eta_{l} }{n \times \prod_{\kappa \neq l}^{n} \eta_{l}^{k_{\kappa}} e^{[\sum_{i=1}^{m} b_{i} j_{\nu_{i}}(g_{i_{0}} \eta_{l}) - |\eta_{l}|^{2}]^{t}} \prod_{l=1}^{n} j_{\nu_{l}}(y \eta_{l}) d\eta_{l} }\]

Thus the second term of sum (6) also vanishes on \( \mathbb{R}^{m+n+1} \times (0, +\infty) \) therefore \( \mathcal{E}(x, t) \) formally satisfies equation (1).

Since for all \( l, r, \)

\[ |B_{k_{i},y_{l}} \mathcal{E}_{2}| \leq \text{const} \prod_{l=1}^{n} \int_{0}^{\infty} \eta_{l}^{k_{r}} e^{-\eta_{l}^{2} t} d\eta_{l} \int_{0}^{\infty} \eta_{l}^{k_{l}+2} e^{-\eta_{l}^{2} t} d\eta_{l}, \]

\[ |T_{y_{l}}^{y_{r}} \mathcal{E}_{2}| \leq \text{const} \prod_{l=1}^{n} \int_{0}^{\infty} \eta_{l}^{k_{l}} e^{-\eta_{l}^{2} t} d\eta_{l}, \]

then \( \left| \frac{\partial \mathcal{E}_{2}}{\partial t} \right| \leq \text{const} \ t^{-\frac{n}{2} - \frac{1}{2} \sum_{i=1}^{m} k_{i}}. \) On the same way \( \left| \frac{\partial \mathcal{E}_{1}}{\partial t} \right| \leq \text{const} \ t^{-\frac{n}{2} - \frac{1}{2} \sum_{i=1}^{m} k_{i}} \) therefore formal differentiating and generalized translation inside integral are legible (at all terms of equation (1)) hence function \( \mathcal{E} \) satisfies equation (1) in \( \mathbb{R}^{m+n+1} \times (0, +\infty) \).

\( \mathcal{E}(x, t) \) will be called fundamental solution of equation (1). Legitimacy of that term will be justified below - we will show that the trace of generalized convolution (see [5], §1.8) of \( \mathcal{E} \) with a bounded initial-value function on the initial hyperplane is the initial-value function itself.

4. Generalized convolution of fundamental solution with bounded functions. On \( \mathbb{R}^{m+n+1} \times (0, +\infty) \) we consider function

\[ \int_{\mathbb{R}^{m+n+1}} \prod_{l=1}^{n} \eta_{l}^{k_{l}} \mathcal{E}(\xi, \eta, t) T_{y_{0}} \mathcal{E} \mathcal{E}_{0}(x - \xi, y) d\xi d\eta. \]

The following assertion is true:
THEOREM 1. Function (7) satisfies equation (1).

Proof. First of all let us prove that function (7) is well-defined. For that purpose we use the following estimates, established for functions (4) and (5) in [9] and [10] correspondingly:

\[ |x|^{m+2} |E_1(x, t)| \leq C \forall t > 0, \]

(8)

\[ y_t^\alpha \int_0^\infty \eta_l^{k_i} e^{-t \left[ \eta_l^2 - \sum_{r=1}^n b_r j_{\alpha}(g_r \eta_l) \right]} j_{\alpha}(y_l \eta_l) d\eta_l \leq C \forall t > 0, \alpha > 0, l \in 1, n. \]

(9)

Here for all positive \( t_0 \) and \( T \) constants of inequalities (8) and (9) depend only on \( t_0 \) and \( T \), but do not depend on \( t \in [t_0, T] \). This and boundedness of function \( u_0 \) yields that integral (7) converges absolutely and uniformly with respect to \( t \in [t_0, T] \) for any bounded \( T \). Really,

\[ \int_{\mathbb{R}_{m+n}^+} \prod_{l=1}^n \eta_l^{k_i} |E(\xi, \eta, t) T^\eta u_0(x - \xi, y)| d\xi d\eta \leq \]

(10)

\[ \leq \frac{1}{2} \sup |u_0| \int_{\mathbb{R}_{m+n}^+} \prod_{l=1}^n |\eta_l|^{k_i} |E(\xi, \eta, t)| d\xi d\eta. \]

In the right-hand side of the latter inequality the integrand is continued to the whole \( \mathbb{R}_{m+n}^+ \) as even with respect to each \( y_l \) function while the inequality itself is understood in the following sense: if its right-hand side converges then its left-hand side converges either, and the inequality itself is valid; note that the normalized Bessel function is even and the generalized translation operator is continuous (see e. g. [5], 18-19).

By the virtue of the smoothness of factors of function \( E(\xi, \eta, t) \) and estimates (8), (9), there exist positive constants \( M_0, \ldots, M_n \) such that the integrand of the latter integral can be represented as \( |f_{0,t}(\xi) \prod_{l=1}^n f_{t,l}(\eta_l)| \) where

\[ |f_{0,t}(\xi)| \leq \frac{M_0}{1 + |\xi|^{m+1}}, \quad |f_{t,l}(\xi)| \leq \frac{M_l}{1 + \eta_l^{k_i}} \quad \text{for } t \in [t_0, T]. \]

Let \( \Omega \) be an arbitrary large bounded domain of \( \mathbb{R}_{m+n}^+ \). Without loss of generality \( Q(1) \subset \Omega \) where \( Q(A) \) denotes \( \{ |\xi| < A, |\eta| \leq A, l = 1, n \} \) for any positive \( A \). There exists \( A_0 \) from \((1, +\infty)\) such that \( \Omega \subset Q(A_0) \).

Function \( |f_{0,t}(\xi) \prod_{l=1}^n f_{t,l}(\eta_l)| \) is integrable over \( Q(A_0) \) by the virtue of the
boundedness of the latter domain therefore Fubini theorem is applicable:

\[ \int_{Q(A_0)} |f_{0,t}(\xi) \prod_{l=1}^{n} f_{l,t}(\eta_l)| d\xi d\eta = \int_{|\xi|<A_0} |f_0(\xi)| d\xi \prod_{l=1}^{n} |f_{l,t}(\eta_l)| d\eta \leq \]

\[ \leq M_0(\max_{l=1,\ldots,n} M_l)^n \left[ \frac{2\pi^m}{m\Gamma(m/2)} + \int_{|\xi|>1} \frac{d\xi}{|\xi|^{m+1}} \right] \left( 2 + 2 \int_1^{A_0} \frac{dz}{z^2} \right)^n \leq \]

\[ \leq M_0(4\max_{l=1,\ldots,n} M_l)^n \left[ \frac{2\pi^m}{m\Gamma(m/2)} + \frac{2\pi^m}{\Gamma(m/2)} \int_1^{\infty} \frac{dr}{r^2} \right] = \frac{2\pi^m M_0}{m\Gamma(m/2)} \left( 4\max_{l=1,\ldots,n} M_l \right)^n (1 + m). \]

Therefore integral at the right-hand side of inequality (10) converges and satisfies the same estimate. It implies that function (7) is well-defined on \( \mathbb{R}^{m+n} \times (0, +\infty) \). Further, by the virtue of self-adjointness of generalized translation operator in the corresponding weighted space (see e.g. [5], p. 19) function (7) is equal to

\[ \int_{\mathbb{R}^{m+n}} \prod_{l=1}^{n} \eta_l^{k_l} u_0(\xi, \eta) T_{y_l}^m \mathcal{E}(x - \xi, y, t) d\xi d\eta. \]

In order to complete the proof it is now left to prove that differentiation and generalized translation inside integral (7) are legible. To do this one has to estimate the behaviour of \( \Delta_x \mathcal{E}, B_{k_l,y_l} \mathcal{E} \) and \( T_{y_l}^{m+r} \mathcal{E} \) at infinity. \( |T_{y_l}^{m+r} \mathcal{E}| = |j_{\nu_l}(g_{l\nu_l}y_l)\mathcal{E}| \leq |\mathcal{E}|. \)

On the other hand, it is proved in [9] that \( |x|^{m+1} \Delta_x \mathcal{E}_1(x, t) \xrightarrow{|x|\to\infty} 0 \) for any positive \( t \), and it is proved in [10] that for all \( t > 0, \alpha > 0, l \in 1, n, \)

\[ y_l^{\alpha} B_{k_l,y_l} \int_0^{\infty} \eta_l^{k_l} e^{-t} \left[ \eta_l^{\alpha} - \sum_{r=1}^{n} b_{l,r} j_{\nu_l}(g_{l\nu_l}) \right] j_{\nu_l}(y_l \eta_l) d\eta_l \xrightarrow{y_l\to\infty} 0. \]

This and (8),(9) imply (as above) that differentiation and generalized translation inside integral (7) are legible. \( \Box \)

5. Solution of the non-classical Cauchy problem. Let us introduce the following denotation:

\[ u(x, y, t) \overset{\text{def}}{=} \frac{2^{n-m}}{\pi^m \prod_{l=1}^{n} 2^{k_l} \Gamma^2 \left( \frac{k_l + 1}{2} \right)} \int_{\mathbb{R}^{m+n}} \prod_{l=1}^{n} \eta_l^{k_l} u_0(x - \xi, \eta) T_{y_l}^m \mathcal{E}(\xi, y, t) d\xi d\eta. \]

The following assertion is true:
THEOREM 2. Function (11) is a solution of problem (1)-(3).

Proof. It follows from Th. 1 that \( u(x, y, t) \) satisfies equation (1), and by the virtue of the evenness of function \( T^y_\eta E(\xi, y, t) \) with respect to variables \( y_1, \ldots, y_n \) (see e. g. [5], p. 35) \( u(x, y, t) \) satisfies evenness condition (2). It is left to prove that it satisfies initial-value condition (3) too.

Let us take arbitrary \( (x_0, y_0) \equiv (x_0^0, \ldots, x_m^0, y_1^0, \ldots, y_n^0) \) from \( \mathbf{R}^{m+n}_+ \) and investigate the behaviour of function \( u(x_0, y_0, t) \) as \( t \to +0 \).

Denoting the constant \( \frac{n!}{\pi^m \prod_{l=1}^{n} 2^k l! \Gamma^2 \left( \frac{k_l}{2} + 1 \right)} \) as \( C \) and taking into account

\[ T^y_\eta f(y) = T^y_\eta f(\eta) \]  

(see e. g. [5], p. 19), we obtain that

\[ u(x_0, y_0, t) = C \int_{\mathbf{R}^{m+n}_+} \prod_{l=1}^{n} \eta^{k_l}_l u_0(x_0 - \xi, \eta) T^y_\eta E(\xi, \eta, t) d\xi d\eta. \]

By means of change of variables \( \zeta_i = \frac{\xi_i}{2\sqrt{t}} \) \( (i = 1, m) \), \( \rho_l = \frac{\eta_l}{2\sqrt{t}} \) \( (l = 1, n) \) the latter equality is reduced to the following form:

\[ u(x_0, y_0, t) = 2^{m+n+|k|} \pi^{\frac{n}{2}} \int_{\mathbf{R}^{m+n}_+} \prod_{l=1}^{n} \rho^{k_l}_l u_0(x_0 - 2\zeta \sqrt{t}, 2\rho \sqrt{t}) T^y_\eta E(2\zeta \sqrt{t}, 2\rho \sqrt{t}, t) d\zeta d\rho \]

(here \( |k| \equiv k_1 + \ldots + k_n \) is the length of multiindex).

Assuming (without loss of generality) that \( m_i = n_i = 1 \) for \( l = 1, n \), \( i = 1, m \), we rename \( b_{i1}, g_{i1}, a_{i1} \) as \( b_i, g_i, a_i \) correspondingly. \( h_i \) will denote \(|h_{i1}|\) if vector \( h_{i1} \) coincides with the positive direction of the \( i \)th co-ordinate axis of space \( \mathbf{R}^m \), and \(-|h_{i1}|\) otherwise \( (i = 1, m) \). Then (see [9])

\[ E_1(x, t) = 2^m \prod_{i=1}^{m} \int_{0}^{+\infty} e^{-t(\tau^2 - a_i \cos h_i \tau)} \cos(x_i \tau + a_i t \sin h_i \tau) d\tau \]

\[ E(2\zeta \sqrt{t}, 2\rho \sqrt{t}, t) = 2^m \prod_{i=1}^{m} \int_{0}^{+\infty} e^{-t(\tau^2 - a_i \cos h_i \tau)} \cos(2\zeta \sqrt{t} \tau + a_i t \sin h_i \tau) d\tau \times \]

\[ \times \prod_{l=1}^{n} \int_{0}^{\eta_l} e^{-t[\eta_l^2 - h_{i1} g_{i1} \eta_l]} \sin(h_{i1} \eta_l) d\eta_l. \]
Substitution $\tau \sqrt{t} = z, \eta \sqrt{t} = \xi (l = \overline{1, n})$ reduces the latter expression to the form

$$
\frac{2^m}{t^{m+n+|k|/2}} \prod_{i=1}^{m} \int_{0}^{+\infty} e^{-z^2 + a_i t \cos \frac{h_i z}{\sqrt{t}}} \cos \left(2z \zeta_i + a_i t \sin \frac{h_i z}{\sqrt{t}}\right) dz \times \prod_{l=1}^{n} \int_{0}^{\xi_{k_l} e^{-\xi^2 + b_l t j_{l_0}} \left(\frac{2 \xi}{\sqrt{t}}\right) j_{\nu_l}(2\xi \rho_l) d\xi.
$$

Thus, using the self-ajointness of generalized translation operator, we obtain:

$$
u(x_0, y_0, t) = 2^{2m+n+|k|} C \int_{R_+^{m+n}} T_{y_0}^{2p} u_0(x_0 - 2\zeta \sqrt{t}, y_0) \times \prod_{i=1}^{m} \int_{0}^{+\infty} e^{-z^2 + a_i t \cos \frac{h_i z}{\sqrt{t}}} \cos \left(2z \zeta_i + a_i t \sin \frac{h_i z}{\sqrt{t}}\right) dz \times \prod_{l=1}^{n} \rho_l^{k_l} \int_{0}^{\xi_{k_l} e^{-\xi^2 + b_l t j_{l_0}} \left(\frac{2 \xi}{\sqrt{t}}\right) j_{\nu_l}(2\xi \rho_l) d\xi d\rho.
$$

Further we will use the following assertions:

**Lemma 1.**

$$
\prod_{i=1}^{m} \int_{0}^{+\infty} e^{-z^2 + a_i t \cos \frac{h_i z}{\sqrt{t}}} \cos \left(2z \zeta_i + a_i t \sin \frac{h_i z}{\sqrt{t}}\right) dz \xrightarrow{t \to +0} \left(\frac{\sqrt{\pi}}{2}\right)^m e^{-|\zeta|^2} \text{ uniformly with respect to } \zeta \in R^m.
$$

**Lemma 2.** For any $i \in \overline{1, m}$, for any positive $A$ there exists $M_i$ depending only on $a_i, h_i$, such that for any $t$ from $(0, 1)$, for any $\zeta_i$ from $(A, +\infty)$

$$
\left| \int_{0}^{+\infty} e^{-z^2 + a_i t \cos \frac{h_i z}{\sqrt{t}}} \cos \left(2z \zeta_i + a_i t \sin \frac{h_i z}{\sqrt{t}}\right) dz \right| \leq \frac{M_i}{\zeta_i^2}.
$$

**Lemma 3.** For any $l \in \overline{1, n}$

$$
\int_{0}^{\xi_{k_l} e^{-\xi^2 + b_l t j_{l_0}} \left(\frac{2 \xi}{\sqrt{t}}\right) j_{\nu_l}(2\xi \rho_l) d\xi \xrightarrow{t \to +0} \frac{\Gamma(\nu_l + 1)}{2} e^{-\rho_l^2} \text{ uniformly with respect to } \rho_l \geq 0.
$$

**Lemma 4.** For any $l \in \overline{1, n}$ there exist $C_l > 0, \alpha > 1$, such that for any $t \in (0, 1)$, for any $\rho_l > 0$

$$
\left| \int_{0}^{\xi_{k_l} e^{-\xi^2 + b_l t j_{l_0}} \left(\frac{2 \xi}{\sqrt{t}}\right) j_{\nu_l}(2\xi \rho_l) d\xi \right| \leq \frac{C_l}{\rho_l^\alpha}.
$$
Lemmas 1, 2 directly follow from Lemmas 5, 4 of [9] respectively, Lemmas 3, 4 are proved in [10].

\[ \int_0^\infty \xi^k e^{-\xi^2} j_n(2\xi\rho_l)d\xi = \frac{\Gamma(\nu_l + 1)}{2} e^{-\rho_l^2} \] (see e. g. [13]) hence

\[ u_0(x_0, y_0) = \frac{2^{m+2n}}{\pi^m \prod_{l=1}^n \Gamma^2(\nu_l + 1)} \int_{\mathbb{R}^{m+n}_+} u_0(x_0, y_0) \prod_{l=1}^m \int_0^\infty e^{-z^2} \cos 2z\xi_l dz \times \]

\[ \times \prod_{l=1}^n \rho_l^{k_l} \int_0^\infty \xi^{k_l} e^{-\xi^2} j_n(2\xi\rho_l) d\xi d\zeta d\rho. \]

Now let us consider the difference \( u(x_0, y_0, t) - u_0(x_0, y_0) \); it equals

\[ 2^{2m+n+|k|C} \int_{\mathbb{R}^{m+n}_+} \prod_{l=1}^n \rho_l^{k_l} \left[ \frac{\mathcal{T}_y^{2\sqrt{t}}}{\gamma_0} u_0(x_0, y_0, t) \times \right. \]

\[ \times \prod_{l=1}^m \int_0^\infty e^{-z^2 + a_l t \cos \frac{h_l t}{\sqrt{t}}} \cos \left( 2z\xi_l + a_l t \sin \frac{h_l t}{\sqrt{t}} \right) dz \times \]

\[ \times \prod_{l=1}^n \int_0^\infty \xi^{k_l} e^{-\xi^2 + b_l t j_n(\frac{h_l t}{\sqrt{t}})} j_n(2\xi\rho_l) d\xi d\zeta d\rho - \]

\[ - u_0(x_0, y_0) \prod_{l=1}^m \int_0^\infty e^{-z^2} \cos 2z\xi_l dz \prod_{l=1}^n \int_0^\infty \xi^{k_l} e^{-\xi^2} j_n(2\xi\rho_l) d\xi \]

\[ d\zeta d\rho. \]

Now we decompose (14) into the following two terms:

\[ 2^{2m+n+|k|C} \left( \int_{Q(A)} + \int_{\mathbb{R}^{m+n}_+ \setminus Q(A)} \right) \overset{\text{def}}{=} 2^{2m+n+|k|C}(I_1 + I_2), \]

where \( A \) is a positive parameter while \( Q(A) \) will hereafter denote the domain \( \{(\zeta, \rho) \in \mathbb{R}^{m+n}_+ | |\zeta| < 1, \rho_l < A; l = 1, n\} \).

Let us take an arbitrary positive \( \varepsilon \) and fix it.

Integral (14) converges absolutely and uniformly with respect to \( t \in (0, 1) \). Really, by the virtue of Lemmas 2, 4 and boundedness of function
$u_0(x, y)$ the absolute value of its integrand is estimated from above (for all $A$ from $(0, +\infty)$ and all $t$ from $(0, 1)$) via

$$\sup |u_0| \left[ \prod_{i=1}^{m} f_{1,i}(\zeta_i) \prod_{l=1}^{n} f_{2,l}(\rho_l) + \frac{2\pi}{2m+n} \prod_{l=1}^{n} \Gamma(\nu_l + 1) e^{-|\zeta|^2 - |\rho|^2} \right],$$

where $0 \leq f_{1,i}(\zeta_i) \leq \frac{M_i}{1 + \zeta_i^2}$, $0 \leq f_{2,l}(\rho_l) \leq \frac{C_i}{1 + \rho_l^2}$, $i = 1, m$, $l = 1, n$. Therefore one can choose a positive $A$ such that $|I_2| < \frac{\epsilon}{2m+n+k+1} C$ for any $t$ from $(0, 1)$. Fix chosen $A$ and consider $I_1$.

$$I_{y_0}^{2\rho\sqrt{t}} u_0(x_0 - 2\zeta \sqrt{t}, y_0) = -u_0(x_0, y_0) =$$

$$= \pi^{-\frac{n}{2}} \prod_{l=1}^{n} \Gamma(\nu_l + 1) \prod_{l=1}^{n} \Gamma(\nu_l + \frac{1}{2}) \int_{0}^{\pi} \int_{0}^{\pi} u_0(x_1^0 - 2\zeta_1 \sqrt{t}, \ldots, x_m^0 - 2\zeta_m \sqrt{t},$$

$$\sqrt{(y_1^0)^2 + 4\rho_1^2 t - 4y_1^0 \rho_1 \sqrt{t} \cos \theta_1}, \ldots, \sqrt{(y_n^0)^2 + 4\rho_n^2 t - 4y_n^0 \rho_n \sqrt{t} \cos \theta_n} \right) \times$$

$$\times \prod_{l=1}^{n} \sin^{2\nu_l} \theta_l \theta_l = -u_0(x_0, y_0) \prod_{l=1}^{n} \sin^{2\nu_l} \theta_l \theta_l =$$

$$= \pi^{-\frac{n}{2}} \prod_{l=1}^{n} \Gamma(\nu_l + 1) \prod_{l=1}^{n} \Gamma(\nu_l + \frac{1}{2}) \int_{0}^{\pi} \int_{0}^{\pi} \left( u_0(x_1^0 - 2\zeta_1 \sqrt{t}, \ldots, x_m^0 - 2\zeta_m \sqrt{t},$$

$$\sqrt{(y_1^0)^2 + 4\rho_1^2 t - 4y_1^0 \rho_1 \sqrt{t} \cos \theta_1}, \ldots, \sqrt{(y_n^0)^2 + 4\rho_n^2 t - 4y_n^0 \rho_n \sqrt{t} \cos \theta_n} \right) -$$

$$-u_0(x_0, y_0) \prod_{l=1}^{n} \sin^{2\nu_l} \theta_l \theta_l.$$
Let \( \delta > 0 \). By the virtue of continuity of function \( u_0 \) at \((x_0, y_0)\) it is possible to choose \( t_0 \) so small that for all \( t \) from \((0, t_0)\), \((\zeta, \rho) \) from \( Q(A) \), \( \theta_l \) from \([0, \pi] \) \((l = 1, \ldots, n)\)

\[
|u_0 \left[ x_1^0 - 2\zeta_1 \sqrt{t}, \ldots, x_m^0 - 2\zeta_m \sqrt{t}, \sqrt{(y_1^0)^2 + 4\rho_1^2 t - 4y_1^0 \rho_1 \sqrt{t} \cos \theta_1}, \ldots \right. \\
\ldots, \sqrt{(y_n^0)^2 + 4\rho_n^2 t - 4y_n^0 \rho_n \sqrt{t} \cos \theta_n} - u_0(x_0, y_0)| < \delta.
\]

It means (since positive \( \delta \) is chosen arbitrary) that

\[
T_{y_0}^{2\rho \sqrt{t}} u_0(x_0 - 2\zeta \sqrt{t}, y_0) \xrightarrow{t \to +0} u_0(x_0, y_0)
\]

uniformly with respect to \((\zeta, \rho) \in Q(A)\). This and Lemmas 1,3 imply that there exists a positive \( t_0 \) such that for any \( t \) from \((0, t_0)\), for any \((\zeta, \rho) \) from \( Q(A) \)

\[
\left| T_{y_0}^{2\rho \sqrt{t}} u_0(x_0 - 2\zeta \sqrt{t}, y_0) \prod_{i=1}^{m} \int_{0}^{\infty} e^{-z^2 + a_i t \cos \frac{h_i z}{\sqrt{t}}} \cos \left(2z\zeta_i + a_i t \sin \frac{h_i z}{\sqrt{t}}\right) dx \times \\
\prod_{l=1}^{n} \int_{0}^{\infty} \zeta_i e^{-\xi^2 + h_l t \nu_l} \left(\frac{\nu_l}{\sqrt{t}}\right) j_{\nu_l}(2\xi \rho_l) d\xi - u_0(x_0, y_0) \prod_{i=1}^{m} \int_{0}^{\infty} e^{-z^2 \cos 2z\zeta_i} dx \times \\
\left. \prod_{l=1}^{n} \int_{0}^{\infty} \xi_i e^{-\xi^2} j_{\nu_l}(2\xi \rho_l) d\xi \right| < \frac{m \Gamma \left(\frac{m}{2}\right) \prod_{l=1}^{n} (k_i + 1)}{\pi \frac{m}{2} A^{m+n+|k|} C^{2m+n+|k|+1}} \xi
\]

that is \( |I_1| \leq \frac{\pi \frac{m}{2} A^{m+n+|k|}}{2^{2m+n+|k|+1} C \pi \Gamma \left(\frac{m}{2}\right) \prod_{l=1}^{n} (k_i + 1) \Gamma \left(\frac{m}{2}\right) \prod_{l=1}^{n} (k_i + 1) Q(A)} \)

Since positive \( \varepsilon \) was chosen arbitrary then \( u(x_0, y_0, t) - u_0(x_0, y_0) \xrightarrow{t \to +0} 0 \).

Thus, since \((x_0, y_0) \) from \( R^m_{+} \) was chosen arbitrary then function \( u(x, y, t) \) satisfies condition (3). \( \Box \)

In particular, with the aid of the latter theorem one can compute the weighted integral of fundamental solution over whole \( R^m_{+} \):
** Lemma 5.**

\[
\int_{\mathbb{R}^{m+n}^+} \prod_{l=1}^{n} y^{k_l} \mathcal{E}(x, y, t) \, dxdy = \frac{\pi^m \prod_{l=1}^{n} \Gamma^2 \left( \frac{k_l + 1}{2} \right)}{2^{n-m-|k|}} e^{t \left( \sum_{i=1}^{m} \sum_{s=1}^{m_i} a_{is} + \sum_{l=1}^{n} \sum_{r=1}^{n_l} b_{lr} \right)}.
\]

**Proof.** Consider \( u_0(x, y) \equiv 1 \); it is continuous and bounded therefore by the virtue of Th. 2 function

\[
y(x, y, t) \overset{\text{def}}{=} \frac{2^{n-m}}{\pi^m \prod_{l=1}^{n} \Gamma^2 \left( \frac{k_l + 1}{2} \right)} \int_{\mathbb{R}^{m+n}^+} \prod_{l=1}^{n} \eta_l^{k_l} \mathcal{E}(\xi, \eta, t) \, d\xi d\eta
\]
satisfies problem (1)-(3) with the initial-value condition \( y(x, y, 0) \equiv 1 \); however \( y(x, y, t) \) does not depend on \( x, y \), therefore \( y(t) \) actually satisfies ordinary differential equation \( y' - y \left( \sum_{i=1}^{m} \sum_{s=1}^{m_i} a_{is} + \sum_{l=1}^{n} \sum_{r=1}^{n_l} b_{lr} \right) = 0 \) and initial-value condition \( y(0) = 1 \). Hence, \( y(t) = e^{t \left( \sum_{i=1}^{m} \sum_{s=1}^{m_i} a_{is} + \sum_{l=1}^{n} \sum_{r=1}^{n_l} b_{lr} \right)} \). □

6. **Case of non-zero right-hand side.** Now we will consider that right-hand side of equation (1) is different from the identical zero; in this case we rename equation (1) as (15). Let us show that with the aid of fundamental solution \( \mathcal{E}(x, y, t) \) it is possible to obtain integral representation of the solution of problem (15)-(2)-(3) too.

For that purpose we fix an arbitrary \((x_0, y_0)\) from \( \mathbb{R}^{m+n}^+ \) and define function

\[
G(t, \tau) \overset{\text{def}}{=} 2^{-2n-m-|k|} \int_{\mathbb{R}^{m+n}^+} \prod_{l=1}^{n} \eta_l^{k_l} f(\xi, \eta, t - \tau) T_{y_0}^{\eta} \mathcal{E}(x_0 - \xi, y_0, \tau) \, d\xi d\eta
\]

for \( t > \tau > 0 \).

The following assertion is true:

**Lemma 6.** There exists a positive \( t_0 \) such that \( G(t, \tau) \) is bounded in \((0, t_0) \times (0, t)\).

**Proof.** Using substitution \( \zeta_i = \frac{\xi_i}{2\sqrt{t}}, \rho_l = \frac{\eta_l}{2\sqrt{\tau}} \) \((i = \overline{1, m}, l = \overline{1, n})\) and self-ajointness of generalized translation operator we obtain that
$G(t, \tau) =$

$$= 2^{-m} t^{m+n+1} \int_{\mathbb{R}^{m+n+1}} \prod_{i=1}^{n} \rho_i^k T_{y_0}^{2\rho \sqrt{\tau}} f(x_0 - 2\zeta \sqrt{\tau}, y_0, t - \tau) \mathcal{E}(2\zeta \sqrt{\tau}, 2\rho \sqrt{\tau}, \tau) d\zeta d\rho.$$ 

Now, as in the proof of Th. 2, we will consider (without loss of generality) that $m_i = n_i = 1$ ($i = 1, m, l = 1, n$). We also rename $h_{1i}, g_{11}, a_{11}$ as $h_i, g_i, a_i$ correspondingly ($i = 1, m, l = 1, n$). Finally, $h_i$ will denote $|h_{1i}|$ if vector $h_{1i}$ coincides with the positive direction of the $i$th co-ordinate axis of space $\mathbb{R}^m$, and $-|h_{1i}|$ otherwise ($i = 1, m$). Then $\mathcal{E}(2\zeta \sqrt{t}, 2\rho \sqrt{t}, t)$ equals to expression (12) therefore

$$G(t, \tau) = \int_{\mathbb{R}^{m+n+1}} \prod_{i=1}^{n} \int_{0}^{+\infty} e^{-z^2 + a_i \tau \cos \frac{h_{1i} z}{\sqrt{T}}} \cos \left(2z\zeta_i + a_i \tau \sin \frac{h_{1i} z}{\sqrt{T}}\right) dz \times$$

$$\times \prod_{i=1}^{n} \rho_i^k \int_{0}^{+\infty} \xi_{k_i} e^{-\xi^2 + b_i \tau j_{n_i} \left(\frac{g_{i1}}{\sqrt{T}}\right)} j_{n_i}(2\xi \rho_i) d\xi T_{y_0}^{2\rho \sqrt{\tau}} f(x_0 - 2\zeta \sqrt{\tau}, y_0, t - \tau) d\zeta d\rho.$$ 

By the virtue of the boundedness of $f$ the last integral converges absolutely and uniformly in the triangle $\{0 < \tau < t < 1\}$ (the proof is identical to the proof of absolute and uniform convergence of the first term of integral (14)) therefore

$$|G(t, \tau)| \leq 2^{-n} \sup |f| \prod_{i=1}^{m} \int_{-\infty}^{+\infty} e^{-z^2 + a_i \tau \cos \frac{h_{1i} z}{\sqrt{T}}} \cos \left(2z\zeta_i + a_i \tau \sin \frac{h_{1i} z}{\sqrt{T}}\right) dz d\zeta_i \times$$

$$\times \prod_{i=1}^{n} \int_{-\infty}^{+\infty} |\rho_i|^k \int_{0}^{+\infty} \xi_{k_i} e^{-\xi^2 + b_i \tau j_{n_i} \left(\frac{g_{i1}}{\sqrt{T}}\right)} j_{n_i}(2\xi \rho_i) d\xi d\rho_i.$$ 

Each external (one-dimensional) integral of the latter expression (i.e. integral over the whole real axis) may be represented as $\int_{-\infty}^{-1} + \int_{-1}^{1} + \int_{1}^{+\infty}$. Further, taking into account the boundedness of internal integrals and applying L. 2 and L. 4 (here we assign $A = 1$ and without loss of generality we consider that $t < 1$), we obtain that there exists $\alpha > 1$ such that

$$|G(t, \tau)| \leq \text{const} \left(2 + 2 \int_{1}^{+\infty} \frac{dr}{r^2} \right)^m \left(2 + 2 \int_{1}^{+\infty} \frac{dr}{r^\alpha} \right)^n.$$
Thus the following function is defined on $\mathbb{R}^{m+n}_+ \times (0, +\infty)$:

\[
v(x, y, t) = \frac{2^{n-m-|k|}}{\pi^m \prod_{l=1}^{n} \Gamma^2\left(\frac{k_l + 1}{2}\right)} \times \int_0^t \int_{\mathbb{R}^{m+n}_+} \prod_{l=1}^{n} \eta_l^{k_l} E(\xi, \eta, \tau) T_{y_0}^\eta f(x - \xi, y, t - \tau) d\xi d\eta d\tau.
\]

By the virtue of evenness of function $T_{y_0}^\eta E(\xi, y, t)$ with respect to variables $y_1, \ldots, y_n$, function (16) satisfies condition (2). Let us prove that it also satisfies equation (15) and homogeneous initial-value condition.

To prove the first of those assertions we note that, as proved in Section 2, function $E(x, y, t)$ satisfies equation (1) in $\mathbb{R}^{m+n}_+ \times (0, +\infty)$. Taking into account estimates of decay at $|x| \to \infty, |y| \to \infty$ for its factors and their derivatives of the corresponding orders, found in Section 3, we obtain that it is left to prove the following lemma:

**Lemma 7.** Let $(x_0, y_0) \in \mathbb{R}^{m+n}_+, t_0 > 0$. Then

\[
\lim_{\tau \to 0} \frac{2^{n-m-|k|}}{\pi^m \prod_{l=1}^{n} \Gamma^2\left(\frac{k_l + 1}{2}\right)} \int_{\mathbb{R}^{m+n}_+} \prod_{l=1}^{n} \eta_l^{k_l} f(x_0 - \xi, y_0, \tau) d\xi d\eta = f(x_0, y_0, t_0).
\]

**Proof.**

\[
\int_{\mathbb{R}^{m+n}_+} \prod_{l=1}^{n} \eta_l^{k_l} f(x_0 - \xi, y_0, \tau) d\xi d\eta = 2^{2m+n+|k|} G(t_0, \tau) =
\]

\[
= 2^{2m+n+|k|} \int \prod_{i=1}^{m} \int_0^{+\infty} e^{-z^2 + a_i \tau \cos \frac{h_i z}{\sqrt{\tau}} + \alpha_i \tau \sin \frac{h_i z}{\sqrt{\tau}}} \cos \left(2z\zeta_i + \alpha_i \tau \sin \frac{h_i z}{\sqrt{\tau}}\right) dz \times
\]

\[
\times \prod_{l=1}^{n} \int_0^{+\infty} e^{-z^2 + b_l \tau \jmath \psi_l(\frac{h_l}{\sqrt{\tau}})} j_{\nu_l}(2\xi \rho_l) d\xi T_{y_0}^{2\rho \sqrt{\tau}} f(x_0 - 2\zeta \sqrt{\tau}, y_0, t_0 - \tau) d\zeta d\rho.
\]
(see proof of L. 6).

It implies that

\[
\frac{2^n - m - |k|}{n \prod_{l=1}^{n} \Gamma^2 \left( \frac{k_l + 1}{2} \right)} \int \prod_{l=1}^{n} \eta_l^{k_l} f(\xi, \eta, t_0 - \tau) \Theta_{1+y} \left( x_0 - \xi, y_0, \tau \right) d\xi d\eta
\]

\[-f(x_0, y_0, t_0) =
\]

\[
= \frac{2^{m+2n}}{\pi^m \prod_{l=1}^{n} \Gamma^2 \left( \frac{k_l + 1}{2} \right)} \int \prod_{l=1}^{n} \rho_l^{k_l} \left[ T_{2\rho}^{2\sqrt{r}} f(x_0 - 2\zeta \sqrt{\tau}, y_0, t_0 - \tau) \times
\]

\[
\times \prod_{i=1}^{m+\infty} \int e^{-z^2 + a_i \tau \cos \frac{h_i}{\sqrt{\tau}} \cos (2\zeta \zeta_i + a_i \tau \sin \frac{h_i\zeta}{\sqrt{\tau}})} dz d\zeta
\]

(17)

\[
\times \prod_{i=1}^{n+\infty} \int \xi_k e^{-\xi^2 + b_i \tau \nu_i \left( \frac{h_i}{\sqrt{\tau}} \right) \nu_i (2\xi \rho_i)} d\xi
\]

\[
= \frac{\pi^{1/2} \prod_{l=1}^{n} \Gamma \left( \frac{k_l + 1}{2} \right)}{2^{m+n}} e^{-|\rho|^2} \int f(x_0, y_0, t_0) d\zeta d\rho
\]

where \(A\) is a positive parameter.

Let \(\varepsilon > 0\).

By the virtue of the boundedness of function \(f\) integral (17) converges absolutely and uniformly with respect to \(\tau \in (0,1)\); the proof is identical to the proof of absolute and uniform convergence of integral (14). Hence one can choose a positive \(A\) such that \(|I_4| < \frac{\varepsilon}{2C}\) for any \(\tau\) from \((0,1)\). Fix chosen \(A\) and consider \(I_3\).

\[
T_{2\rho}^{2\sqrt{r}} f(x_0 - 2\zeta \sqrt{\tau}, y_0, t_0 - \tau) = \pi^{-3/2} \prod_{i=1}^{n} \frac{\Gamma(v_i + 1)}{\Gamma(v_i + \frac{1}{2})} \times
\]

\[
\times \int_{0}^{\pi} \cdots \int_{0}^{\pi} f \left[ x^0_1 - 2\zeta_1 \sqrt{\tau}, \ldots, x^0_m - 2\zeta_m \sqrt{\tau}, \sqrt{(y^0_1)^2 + 4\rho^2 \tau - 4y^0_1 \rho \tau \cos \theta_1}, \ldots
\]

\[
\text{n times}
\]
... \sqrt{g^2_n + 4 \rho^2_n \tau - 4 y^0_n \rho_n \sqrt{\tau} \cos \theta_n}, t_0 - \tau \right) \prod_{l=1}^{n} \sin^{2\nu_l} \theta_l d\theta_l.

By the virtue of continuity and boundedness of \( f \) the latter expression tends to \( f(x_0, y_0, t_0) \) uniformly with respect to \( (\zeta, \rho) \in Q(A) \) as \( \tau \to +0 \). This and Lemmas 1,3 imply that there exists a positive \( \tau_0 \) such that for any \( \tau < \tau_0 \), for any \( \eta \in Q(A) \)

\[
\left| T_{y_0}^{2\rho \sqrt{\tau}} f(x_0 - 2\zeta \sqrt{\tau}, y_0, t_0 - \tau) \prod_{l=1}^{m} e^{-z^2 + a_i \tau \cos \frac{b_i z}{\sqrt{\tau}}} \cos \left( 2z \zeta_i + a_i \tau \sin \frac{h_i z}{\sqrt{\tau}} \right) dz \times \right.

\[
\left. \prod_{l=1}^{m} \int \xi^{k_i} e^{-\xi^2 + h_l \tau j_{n_l} \left( \frac{m_{n_l}}{\rho_l} \right)} j_{n_l} (2\xi \rho_l) d\xi - \frac{\pi^{m} \prod_{l=1}^{n} \Gamma \left( \frac{k_l + 1}{2} \right)}{2^{m+n}} e^{-|\rho|^2} f(x_0, y_0, t_0) \right| < \frac{m \Gamma \left( \frac{m}{2} \right) \prod_{l=1}^{n} (k_l + 1)}{4 \pi^{m} A^{m+n+1}} \varepsilon \quad \text{that is } |I_3| \leq \frac{\varepsilon}{2}.
\]

\[
\text{It is left to prove that } v(x_0, y_0, t) \xrightarrow{t \to 0^+} 0 \text{ for any } (x_0, y_0) \text{ from } \mathbb{R}^{m+n}_+.
\]

To do this we represent \( v(x_0, y_0, t) \) as \( \frac{2^{m+2n}}{\pi^m \prod_{l=1}^{n} \Gamma \left( \frac{k_l + 1}{2} \right)} \int G(t, \tau) d\tau \) and use L. 6: there exists a positive \( t_0 \) such that \( |v(x_0, t)| \leq \frac{2^{m+2n} \sup_{t \in [0, t_0]} |G|}{\pi^m \prod_{l=1}^{n} \Gamma \left( \frac{k_l + 1}{2} \right)} \) for any \( t \) from \( (0, t_0) \). Thus, since point \( (x_0, y_0) \) was chosen arbitrary, the following assertion is proved:

**Theorem 3.** Let \( u_0(x, y) \) be continuous and bounded in \( \mathbb{R}^{m+n}_+ \). Let \( f, \)

\[
\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_m}, \frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_n} \]

be continuous and bounded in \( \mathbb{R}^{m+n}_+ \times (0, +\infty) \). Let \( f \) satisfy condition (2). Then function

\[
\frac{2^{n-m-|k|}}{\pi^m \prod_{l=1}^{n} \Gamma \left( \frac{k_l + 1}{2} \right)} \left[ \int_{\mathbb{R}^{m+n}_+} \prod_{l=1}^{n} \eta^{k_l} \mathcal{E}(\xi, \eta, t) T^\eta_y u_0(x - \xi, y) d\xi d\eta + \int_{0}^{t} \int_{\mathbb{R}^{m+n}_+} \prod_{l=1}^{n} \eta^{k_l} \mathcal{E}(\xi, \eta, \tau) T^\eta_y f(x - \xi, y, t - \tau) d\xi d\eta d\tau \right]
\]

(18)
is a solution of problem (15)-(2)-(3).

7. Uniqueness of the solution. First of all we prove the following assertion:

**Lemma 8.** For any positive $T$ function (18) is bounded in $\mathbb{R}_+^{m+n} \times [0, T]$.

**Proof.** By means of (13) we represent the found solution of problem (15)-(2)-(3) as follows:

$$u(x, y, t) =$$

$$= C_1 \left[ \int_{\mathbb{R}_+^{m+n}} T_y e^{\sqrt{t} \int_0^t u_0(x - 2\sqrt{t} \cdot y, \cdot) \prod_{i=1}^m \int_0^{+\infty} e^{-z^2 + a_i t \cos \frac{b_i z}{\sqrt{t}}} \cos \left(2z \xi_i + a_i t \sin \frac{h_i z}{\sqrt{t}}\right) dz \times \right.$$

$$\times \prod_{l=1}^n \left[ \xi_{l}^2 \int_0^{+\infty} e^{-z^2 + a_l t \cos \frac{b_l z}{\sqrt{t}}} \cos \left(2z \xi_i + a_l t \sin \frac{h_l z}{\sqrt{t}}\right) dz \times \right.$$

$$\left. \prod_{i=1}^m \int_0^{+\infty} e^{-z^2 + a_i t \cos \frac{b_i z}{\sqrt{t}}} \cos \left(2z \xi_i + a_i t \sin \frac{h_i z}{\sqrt{t}}\right) dz \times \right.$$

$$\left. \prod_{l=1}^n \int_0^{+\infty} e^{-z^2 + a_l t \cos \frac{b_l z}{\sqrt{t}}} \cos \left(2z \xi_i + a_l t \sin \frac{h_l z}{\sqrt{t}}\right) dz \right] \text{def} C_1[I_5(x, y, t) + I_6(x, y, t)].$$

Here coefficients of the equation are under the same assumptions as in the proof of Th. 2 (that is no loss of generality occurs) but we took into account that right-hand side of the equation is in general different from the identical zero.

Integrating $\int_0^{+\infty} e^{-z^2 + a_i t \cos \frac{b_i z}{\sqrt{t}}} \cos \left(2z \xi_i + a_i t \sin \frac{h_i z}{\sqrt{t}}\right) dz (i = 1, m)$ by parts two times, we obtain that for $0 \leq t \leq T, \xi_i \neq 0$ modulus of the latter integral does not exceed $\frac{M_t(1 + T) e^{a_i T}}{\xi_i^2}$, where constant $M_t$ depends merely on coefficients of equation (1).

Integrating $\int_0^{+\infty} e^{-z^2 + a_l t \cos \frac{b_l z}{\sqrt{t}}} \cos \left(2z \xi_i + a_l t \sin \frac{h_l z}{\sqrt{t}}\right) dz (l = 1, n)$ by parts $n_0$ times (here $n_0$ is the unique natural number situating in $(\nu_l + \frac{3}{2}, \nu_l + \frac{5}{2})$), we obtain
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(see [10]) that for $0 \leq t \leq T, \rho_l > 0$ modulus of the latter integral does not exceed $\frac{M_l(T)e^{\left|b_l\right|T}}{\rho_l^{k+\alpha}}$, where $M_l$ is a linear combination of power functions with non-negative powers, and its coefficients depend merely on coefficients of equation (1).

Obtained estimates are used for $|\zeta_i| > 1, \rho_l > 1; (i = \overline{1, m}, l = \overline{1, n})$. If $|\zeta_i| \leq 1, \rho_l \leq 1$ then absolute values of the mentioned integrals are evidently estimated from above via $\frac{\sqrt{\pi}e^{\left|a_i\right|T}}{2}, \frac{\Gamma(\nu_l + 1)e^{\left|b_l\right|T}}{2}$ correspondingly.

Using of those estimates and of the boundedness of functions $u_0, f$ concludes the proof. \( \square \)

Now we can apply Fourier transforms method (see [3], Ch. 2, § 4 and Appendix 1) with the using of Fourier-Bessel transformation (see e. g. [5], Ch. 1) to investigate the uniqueness of the found solution of problem (15)-(2)-(3). To do this we introduce below (following [3], Ch. 1) special spaces of test functions (cf. [5], § 1.1). Here we will consider that condition (2) is replaced with the equivalent condition of evenness of function $u$ with respect to each variable $y_l$ ($l = \overline{1, n}$) while the problem itself is respectively considered in the whole $\mathbb{R}^{m+n} \times (0, +\infty)$ (see Section 1).

Let $\mu_i, \omega_i$ be continuous increasing on $[0, +\infty)$ functions such that

$$\lim_{r \to +\infty} \mu_i(r) = \lim_{r \to +\infty} \omega_i(r) = \infty, \mu_i(0) = \omega_i(0) = 0; \ i = \overline{1, m+n}.$$ 

Further, on $[0, +\infty)$ we define the following increasing concave functions:

$$M_i(r) \overset{\text{def}}{=} \int_0^r \mu_i(\rho)d\rho, \ \Omega_i(r) \overset{\text{def}}{=} \int_0^r \omega_i(\rho)d\rho.$$ 

Then the space of test functions $W_M^\Omega \overset{\text{def}}{=} W_{M_1,...,M_{m+n}}^{\Omega_1,...,\Omega_{m+n}}$ will be formed as the set of all even with respect to each $y_l$ ($l = \overline{1, n}$) entire functions of $m + n$ complex variables such that the following estimate holds:

$$|\varphi(x_1, \ldots, x_m, y_1, \ldots, y_n)| \leq \sum_{i=1}^{m} M_i(\alpha_i \Re x_i) - \sum_{i=1}^{m} M_{m+i}(\alpha_i \Re y_i) + \sum_{i=1}^{m} \Omega_i(\beta_i \Re x_i) + \sum_{i=1}^{m} \Omega_{m+i}(\beta_i \Re y_i),$$ 

where constants $C, \alpha_1, \ldots, \alpha_{m+n}, \beta_1, \ldots, \beta_{m+n}$ may depend on the test function $\varphi$.

The topology in $W_M^\Omega$ is introduced on the classical way: we will say that sequence $\{\varphi_\nu\}_{\nu=1}^{\infty}$ converges to zero in $W_M^\Omega$ if it uniformly converges to zero in
each bounded domain of $C^{m+n}$ and constants $C, \alpha_1, \ldots, \alpha_{m+n}, \beta_1, \ldots, \beta_{m+n}$ can be taken independent on $\nu \in \mathbb{N}$.

So set $Q \subset \mathcal{W}^{\Omega}_{\mathcal{M}}$ is called bounded if estimate (19$_{a,b}$) holds for all elements of $Q$ with same constants $C, \alpha_1, \ldots, \alpha_{m+n}, \beta_1, \ldots, \beta_{m+n}$.

Fourier-Bessel transformation is defined on $\mathcal{W}^{\Omega}_{\mathcal{M}}$ as follows:

$$\hat{f}(\xi, \eta) \overset{\text{def}}{=} \mathcal{F}_b f \overset{\text{def}}{=} \int_{\mathbb{R}^{m+n}} \prod_{l=1}^{n} y_l^k j_{\nu_l}(\eta_l y_l) e^{-ix\xi} f(x, y) dx dy.$$

$W^{\Omega}_{\mathcal{M},\alpha}$ denotes the subset of $\mathcal{W}^{\Omega}_{\mathcal{M}}$ such that for each its element inequality (19$_{a,b}$) is valid for any

$$0 < \alpha_1 < \ldots < \alpha_{m+n} < \beta_1 < \beta_2 < \ldots < \beta_{m+n}.$$

The following assertion is true:

**LEMMA 9.** Let for any $i \in 1, m+n$ functions $\hat{M}_i$ and $\hat{\Omega}_i$ are dual in the sense of Young to functions $\Omega_i$ and $M_i$ correspondingly. Then Fourier-Bessel transformation is a bounded operator mapping $W^{\Omega}_{\mathcal{M},\alpha}$ onto $W^{\Omega}_{\mathcal{M},\beta}$, where

$$\frac{1}{\alpha} = \left(\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_{m+n}}\right), \quad \frac{1}{\beta} = \left(\frac{1}{\beta_1}, \ldots, \frac{1}{\beta_{m+n}}\right).$$

Proof. $j_{\nu}(x + iy) = \frac{\Gamma(\nu + 1)}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int e^{(ix-y)t}(1 - t^2)^{\nu - \frac{1}{2}} dt$ for any $\nu > -\frac{1}{2}$ (see e. g. [5], (1.5.8)) hence $|j_{\nu}(x + iy)| \leq \text{const} e^{\nu y}$.

Further proof is entirely identical to the proof of Th. 4 of [3], Ch. 1, §3.

Now we observe elliptic operator $A$ containing at equation (1):

$$Au \overset{\text{def}}{=} \sum_{i=1}^{m} \left[ \sum_{s=1}^{m_1} a_{is} u(x + h_{is}, y, t) \right] + \sum_{l=1}^{m} \left( B_{li} y_l u + \sum_{r=1}^{m_1} b_{lir} T_{y_l}^{\nu_r} u \right).$$

Let us find its symbol $P(z) \overset{\text{def}}{=} P(z_1, \ldots, z_{m+n}) \overset{\text{def}}{=} P(\sigma_1 + i\tau_1, \ldots, \sigma_{m+n} + i\tau_{m+n})$. It is enough to consider the case $m_1 = n_1 = 1$ (i. e. the case of one special and one non-special spatial variables). Then

$$P(z_1, z_2) = -z_1^2 + \sum_{s=1}^{m_1} a_{1s} e^{-ih_{1s}z_1} - z_2^2 + \sum_{r=1}^{n_1} b_{1r} j_{\nu_1}(g_{1r} z_2)$$

(see [5], (1.3.5) and (1.3.8)).

$$\Re P(z) = |\sigma|^2 - |\tau|^2 + \sum_{s=1}^{m_1} a_{1s} e^{h_{1s} \gamma} \cos h_{1s} \sigma_1 + \sum_{r=1}^{n_1} b_{1r} \Re j_{\nu_1}(g_{1r} z_2).$$
We will call $R(\xi, \eta)$ positive definite (cf. [17], § 8) if there exists a positive $C$ such that $\Re R(\xi, \eta) \geq C(|\xi|^2 + |\eta|^2)$ for any $(\xi, \eta)$ from $\mathbb{R}_+^{m+n}$.

Together with equation (1) we consider equation

$$
\frac{\partial w}{\partial t} = \sum_{i=1}^{m} \frac{\partial^2 w}{\partial x_i^2} + \sum_{l=1}^{n} B_{ki} w, \quad (x, y) \in \mathbb{R}_+^{m+n}, \quad t > 0,
$$

and initial-value condition

$$
w\big|_{t=0} = w_0(x, y), \quad (x, y) \in \mathbb{R}_+^{m+n},
$$

where $w_0$ is continuous and bounded.

It is known from [6] (see also [7] and [8]) that problem (21)-(2)-(22) has a unique classical bounded solution $w(x, y, t)$.

The following assertion is valid:

**Theorem 5.** Let $f(x, y) \equiv 0$, $R(\xi, \eta)$ be positive definite. Then for any $(x, y)$ from $\mathbb{R}_+^{m+n}$

$$
e^{-t\left(\sum_{i=1}^{m} \sum_{s=1}^{m_i} a_{is} + \sum_{l=1}^{n} \sum_{r=1}^{n_l} b_{lr}\right)} u(x, y, t) -$$

$$- w\left(\frac{x_1 + q_1 t}{p_1}, \ldots, \frac{x_m + q_m t}{p_m}, \frac{y_1}{p_{m+1}}, \ldots, \frac{y_n}{p_{m+n}}, t\right) \to 0, \quad t \to \infty,
$$

where

$$p_i = \sqrt{1 + \frac{1}{2} \sum_{s=1}^{m_i} a_{is} h_{is}^2}, \quad q_i = \sum_{s=1}^{m_i} a_{is} h_{is}, \quad i = \overline{1, m};$$

$$p_{m+l} = \sqrt{1 + \frac{1}{2(k_l + 1)} \sum_{r=1}^{n_l} b_{lr} g_{lr}^2}, \quad l = \overline{1, n};$$

$$w_0(x, y) = w_0(p_1 x_1, \ldots, p_m x_m, p_{m+1} y_1, \ldots, p_{m+n} y_n).$$

**Proof.** First of all let us prove that $p_1, \ldots, p_{m+n}$ are well-defined and different from the identical zero under the assumptions of the theorem. Without loss of generality $(finite)$ sequences $\{a_{is}\}_{s=1}^{m_i}, \{b_{lr}\}_{r=1}^{n_l}$ are non-increasing for all $i \in \overline{1, m}, \quad l \in \overline{1, n}$. Let us denote $\min_s a_{is} > 0$ as $m_i^0$, and $\min_r b_{lr} > 0$ as $n_l^0$; for those
$i, l$, where all the coefficients are negative, we denote $m_i + 1$ or $n_i + 1$ as $m_i^0$ or $n_i^0$ correspondingly.

Let $i \in \overline{1, m}$. The assumption of the theorem implies that

$$\sum_{s < m_i^0} a_{is} + \xi_i^2 - \sum_{s < m_i^0} a_{is} \cos h_{is} \xi_i \geq C \xi_i^2$$

for any positive $\xi_i$ (by the virtue of the condition of positive definiteness, in which $\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_m, \eta_1, \ldots, \eta_n$ are assigned equal to zero). Then $\frac{1}{2} \sum_{s < m_i^0} a_{is} h_{is}^2 > -1$ hence $p_1, \ldots, p_m$ are well-defined and positive.

Now let $l \in \overline{1, n}$. Then

$$\sum_{r < n_i^0} b_{lr} + \eta_i^2 - \sum_{r < n_i^0} b_{lr} j_{\nu_l}(g_{lr} \eta_i) \geq C \eta_i^2$$

for any positive $\eta_i$, therefore

$$C \eta_i^2 \leq \eta_i^2 + \sum_{r < n_i^0} b_{lr} [1 - j_{\nu_l}(g_{lr} \eta_i)].$$

$$j_{\nu}(z) = 1 - \frac{z^2}{4(\nu + 1)} + O(z^4) \implies 1 - j_{\nu}(z) = \frac{z^2}{4(\nu + 1)} + O(z^4),$$

hence in a certain neighbourhood of the origin

$$C \eta_i^2 \leq \eta_i^2 + \frac{\eta_i^2}{4(\nu_i + 1)} \sum_{r < n_i^0} b_{lr} g_{lr}^2 + O(\eta_i^4) \implies$$

$$C \leq 1 + \frac{1}{4(\nu_i + 1)} \sum_{r < n_i^0} b_{lr} g_{lr}^2 + O(\eta_i^2) \implies$$

$$C \leq 1 + \frac{1}{4(\nu_i + 1)} \sum_{r < n_i^0} b_{lr} g_{lr}^2 + O(\eta_i^2) - \frac{C}{2}$$

i.e. in a small enough neighbourhood of the origin

$$0 < \frac{C}{2} \leq 1 + \frac{1}{4(\nu_i + 1)} \sum_{r < n_i^0} b_{lr} g_{lr}^2 \implies \frac{1}{4(\nu_i + 1)} \sum_{r < n_i^0} b_{lr} g_{lr}^2 > -1.$$
LEMMA 10. Under the assumptions of Th. 5 for any \( l = m + 1, m + n \)

\[
\int_0^\infty \eta^{2\nu+1} e^{-\eta^2 + \frac{\eta^2}{2}} \sum_{r=1}^n \left[ j_{\nu} \left( \frac{2\nu \eta}{\sqrt{\nu}} \right) \right] j_{\nu} \left( 2\rho \eta \right) d\eta \rightarrow \frac{\Gamma(\nu + 1)}{2^\nu \rho^{\nu+1}} e^{-\frac{\rho^2}{\nu}}
\]

uniformly with respect to \( \rho \geq 0 \).

Proof. First of all let us note that the symbol of sum and index \( r \) can be omitted because it is obviously enough to prove the lemma for the case of one term. We will also omit index \( l \) (since it was chosen arbitrary), and rename \( p^2 \) as \( p \). Further,

\[
\int_0^\infty \eta^{2\nu+1} e^{-\rho^2} j_{\nu} \left( 2\rho \eta \right) d\eta = \frac{1}{\rho^{\nu+1}} \frac{\Gamma(\nu + 1)}{2} e^{-\frac{\rho^2}{\nu}},
\]

so it is enough to prove that

\[
\int_0^\infty \eta^{2\nu+1} j_{\nu} \left( 2\rho \eta \right) \left( e^{-\eta^2 + \rho^2} \left[ j_{\nu} \left( \frac{2\nu \eta}{\sqrt{\nu}} \right) \right] - e^{-\rho^2} \right) d\eta \rightarrow 0
\]

uniformly with respect to \( \rho \geq 0 \).

First of all we prove absolute and uniform (with respect to \( t, \rho \)) convergence of the latter integral. Since \( \rho \) is positive and function \( j_{\nu} \) is bounded then it is enough to estimate the power of the first exponent of the integrand for that purpose. Here we consider that \( b < 0 \) because otherwise the proving convergence is evident.

Let us estimate function \( f(z) \equiv z^2 - a [j_{\nu}(z) - 1] \), where \( a \) is a non-negative parameter. \( f'(z) = 2z + a \frac{z}{2\nu + 2} j_{\nu+1}(z) = 2z \left[ 1 + a \frac{1}{4\nu + 4} j_{\nu+1}(z) \right] \geq 0 \) for

\[
\left| \frac{a}{4\nu + 4} \right| \leq 1 \Leftrightarrow a \geq -4\nu - 4.
\]

Thus function \( f \) does not decrease on the half-line \([0, +\infty)\) for \( a \geq -4\nu - 4 \). Since \( f(0) = 0 \) then \( f \) is non-negative on the whole real line (by the virtue of its evenness) for \( a \geq -4\nu - 4 \). Now let \( a > -4\nu - 4 \). Then there exists \( \alpha \) from \((0,1)\) such that \( \frac{a}{1 - \alpha} \geq -4\nu - 4 \). Therefore

\[
f(z) - \alpha z^2 = (1 - \alpha) z^2 - a [j_{\nu}(z) - 1] = (1 - \alpha) \left( z^2 - \frac{a}{1 - \alpha} [j_{\nu}(z) - 1] \right) \geq 0.
\]

Thus for any \( a > -4\nu - 4 \) there exists a positive \( \alpha \) such that \( f(z) \geq \alpha z^2 \) on \( \mathbb{R}^1 \).
Rename $\frac{g\eta}{\sqrt{t}}$ as $z$ in order to represent the estimating power in the following form:

$$\frac{z^2}{g^2} + bt|j_\nu(z) - 1| = -\frac{t}{g^2} (z^2 - bg^2|j_\nu(z) - 1|).$$

Since $bg^2 > -4\nu - 4$ then there exists a positive $\alpha$ such that the latter expression does not exceed $-\frac{t\alpha}{g^2} z^2 = -\alpha \eta^2$ hence the latter integral converges absolutely and uniformly.

Then we set an arbitrary positive $\varepsilon$ and decompose the latter integral into sum $\int^\delta_0 + \int^\infty_\delta \defn I_1 + I_2$. Using have just proved absolute and uniform convergence we choose a positive $\delta$ such that $|I_2| \leq \frac{\varepsilon}{2}$ for any $t \geq 1$. Let us fix the chosen $\delta$ and estimate $I_1$. Its modulus does not exceed

$$\int^\delta_0 \eta^{2\nu+1} e^{-\eta t^2} \left| e^{(p-1)\eta^2 + bt\left[ j_\nu'(\frac{g\eta}{t}) - 1 \right]} \right| d\eta.$$

To estimate the latter expression we write

$$j_\nu(z) = j_\nu(0) + j_\nu'(0)z + \frac{j_\nu''(0)}{2} z^2 + \frac{j_\nu'''(\theta)}{6} z^3,$$

where $\theta \in [0, z]$.

$$j_\nu(0) = 1, \ j_\nu'(0) = 0, \ j_\nu''(0) = -\frac{1}{2\nu + 2},$$

$$j_\nu'''(\theta) = \frac{3\theta j_{\nu+2}(\theta)}{4(\nu + 1)(\nu + 2)} - \frac{\theta^3 j_{\nu+3}(\theta)}{8(\nu + 1)(\nu + 2)(\nu + 3)}.$$ 

Thus $j_\nu\left( \frac{g\eta}{\sqrt{t}} \right) - 1 = -\frac{1}{4(\nu + 1)} \frac{g^2 \eta^2}{t} + \frac{\psi(\eta, t) g^3 \eta^3}{t^3}$, where

$$|\psi(\eta, t)| \leq \frac{3g\delta}{8(\nu + 1)(\nu + 2)} + \frac{(g\delta)^3}{48(\nu + 1)(\nu + 2)(\nu + 3)} \text{ for any } t \geq 1, \eta \leq \delta.$$ 

Therefore the power of the second exponent in the latter integral is equal to

$$\left[ p - 1 - \frac{bg^2}{4(\nu + 1)} \right] \eta^2 + \frac{b\psi(\eta, t) g^3 \eta^3}{\sqrt{t}} = \frac{b\psi(\eta, t) g^3 \eta^3}{\sqrt{t}} \defn \tilde{\psi}(\eta, t)/\sqrt{t},$$
where \( \tilde{\psi}(\eta, t) \) is bounded on \([0, \delta] \times [1, +\infty)\).

Thus \( |I_1| \leq \int_0^{\delta} \eta^{2\nu + 1} e^{-\eta t^2} \left| e^{\frac{2(\eta t)}{\sqrt{t}}} - 1 \right| d\eta. \)

Let us choose \( t_0 \) from \([1, +\infty)\) such that \( e^\frac{M}{\sqrt{t_0}}, e^{-\frac{M}{\sqrt{t_0}} \in [1 - \delta_0, 1 + \delta_0]\),

where \( \delta_0 \) denotes constant \( \frac{\varepsilon}{2} \left( \int_0^{\delta} \eta^{2\nu + 1} e^{-\eta t^2} d\eta \right)^{-1}. \) Then \( |I_1| \leq \frac{\varepsilon}{2} \) for any \( t \geq t_0 \) by the virtue of the monotonicity of the exponent. \( \square \)

Now we are able to pass directly to the proof of relation (23). It is obviously sufficient to do that for the case

\[ m_1 = \ldots = m_m = n_1 = \ldots = n_n = 1 \]

so we will omit the second indices of coefficients \( a, b, h, g. \)

Let \((x_0, y_0) = (x_0^0, \ldots, x_m^0, y_0^0, \ldots, y_n^0)\) be an arbitrary point of \( R_t^{m+n}. \)

Applying formula (13) we represent \( e^{\varepsilon t} u(x_0, y_0, t) \)

\[ \frac{2^m 2^n}{\pi^m \prod_{i=1}^{m} \Gamma^2(\nu_i + 1)} \int_{R_t^{m+n}} T_{y_0}^{2\rho \sqrt{t}} u_0(x_0 - 2\zeta, y_0) \times \]

\[ \prod_{i=1}^{m} \int_0^{+\infty} e^{-x^2 + a_i t \left( \cos \frac{h_i z}{\sqrt{t}} - 1 \right)} \cos \left( 2z\zeta_i + a_i t \sin \frac{h_i z}{\sqrt{t}} \right) dz \times \]

\[ \prod_{l=1}^{n} \int_0^{+\infty} e^{-x^2 + b_l t \left[ j_{\nu_l} ( \frac{y_l}{\sqrt{t}} ) \right]^{-1}} j_{\nu_l} (2\zeta \rho) d\zeta d\rho. \]

Further we will without loss of generality consider that \( m = n = 1. \) Then the latter expression takes the following form:

\[ \frac{8}{\pi \Gamma^2(\frac{\nu + 1}{2})} \int_{-\infty}^{0} \int_{0}^{\infty} \rho^2 T_{y_0}^{2\rho \sqrt{t}} u_0(x_0 - 2\zeta, y_0) \int_0^{+\infty} e^{-x^2 + a t \left( \cos \frac{h z}{\sqrt{t}} - 1 \right)} \times \]

\[ \cos \left( 2z\zeta + a t \sin \frac{h z}{\sqrt{t}} \right) dz \int_0^{+\infty} e^{-x^2 + b t \left[ j_{\nu} ( \frac{y}{\sqrt{t}} ) \right]^{-1}} j_{\nu} (2\zeta \rho) d\zeta d\rho. \]
Together with it we observe

\[
\frac{8}{\pi \Gamma^2 \left( \frac{k+1}{2} \right)} \int_0^{\infty} \int_{-\infty}^{\infty} \rho^{k} T_{y_0}^2 \sqrt{t} u_0(x_0 - 2\zeta \sqrt{t}, y_0) \times
\]
\[
\frac{8}{\pi \Gamma^2 \left( \frac{k+1}{2} \right)} \int_0^{\infty} \int_{-\infty}^{\infty} \rho^{k} T_{y_0}^2 \sqrt{t} u_0(x_0 - 2\zeta \sqrt{t}, y_0) \int_0^{\infty} e^{-\frac{(2\zeta - ah \sqrt{t})^2}{4 \nu_1^2}} \Gamma \left( \frac{k+1}{2} \right) e^{-2\nu_1^2} d\zeta d\rho =
\]
\[
= \frac{8}{\pi \Gamma^2 \left( \frac{k+1}{2} \right)} \int_0^{\infty} \int_{-\infty}^{\infty} \rho^{k} T_{y_0}^2 \sqrt{t} u_0(x_0 - 2\zeta \sqrt{t}, y_0) \int_0^{\infty} e^{-\frac{(2\zeta - ah \sqrt{t})^2}{4 \nu_1^2}} \Gamma \left( \frac{k+1}{2} \right) e^{-2\nu_1^2} d\zeta d\rho =
\]
\[
= \frac{2}{\nu_1 \Gamma \left( \frac{k+1}{2} \right) p_1 p_2} \int_0^{\infty} \int_{-\infty}^{\infty} \rho^{k} T_{y_0}^2 \sqrt{t} u_0(x_0 - 2\zeta \sqrt{t}, y_0) e^{-\frac{(2\zeta + ah \sqrt{t})^2}{4 \nu_1^2}} - \frac{\epsilon^3}{\nu_1^2} d\zeta d\rho.
\]

On the other hand it is known from [6]-[8] that

\[
w(x_0, y_0, t) = \frac{2}{\sqrt{\pi} \Gamma \left( \frac{k+1}{2} \right)} \int_0^{\infty} \int_{-\infty}^{\infty} \rho^{k} e^{-\zeta^2 - \rho^2} T_{y_0}^2 \sqrt{t} u_0(x_0 - 2\zeta \sqrt{t}, y_0) d\zeta d\rho
\]

therefore

\[
w \left( \frac{x_0 + q_1 t}{p_1}, \frac{y_0}{p_2}, t \right) =
\]
\[
= \frac{2}{\sqrt{\pi} \Gamma \left( \frac{k+1}{2} \right) p_1 p_2} \int_0^{\infty} \int_{-\infty}^{\infty} \rho^{k} e^{-\frac{(2\zeta + ah \sqrt{t})^2}{4 \nu_1^2}} - \frac{\epsilon^3}{\nu_1^2} T_{y_0}^2 x_0 - 2\zeta \sqrt{t}, y_0) d\zeta d\rho =
\]
\[
= \frac{2}{\sqrt{\pi} \Gamma \left( \frac{k+1}{2} \right) p_1 p_2} \int_0^{\infty} \int_{-\infty}^{\infty} \rho^{k} T_{y_0}^2 \sqrt{t} u_0(x_0 - 2\zeta \sqrt{t}, y_0) e^{-\frac{(2\zeta + ah \sqrt{t})^2}{4 \nu_1^2}} - \frac{\epsilon^3}{\nu_1^2} d\zeta d\rho.
\]
Thus expression (24) is equal to $w\left(\frac{x_0 + q_1 t}{p_1}, \frac{y_0}{p_2}, t\right)$, i.e. in order to conclude relation (23) one has to investigate behaviour as $t \to \infty$ of the following integral:

$$
\int_0^\infty \int_{-\infty}^\infty \rho^2 T_{yo}^2 u_0(x_0 - 2\zeta \sqrt{t}, y_0) \left[ \int_0^\infty e^{-z^2 + at} \left( \cos \frac{b\zeta}{\sqrt{t}} - 1 \right) \times \right.
$$

$$
\times \cos \left(2z\zeta + at \sin \frac{h\zeta}{\sqrt{t}}\right) dz \int_0^\infty \xi^k e^{-\xi^2 + bt} \left[ j_\nu\left(\frac{\xi}{\sqrt{t}}\right) - 1\right] j_\nu(2\xi \rho) d\xi -
$$

$$
- \frac{\sqrt{\pi}}{2p_1} \int_0^\infty e^{-\frac{\xi^2}{p_1^2}} \frac{\Gamma(\nu + 1)}{2p_2^{\nu + 1}} e^{-\frac{\xi^2}{p_2^2}} d\xi d\rho.
$$

(25) First of all we prove absolute and uniform (with respect to $t \in [1, +\infty)$) convergence of the latter integral. By the virtue of the boundedness of function $u_0$ the absolute value of its second term is estimated from above by

$$
\text{const} \int_0^\infty e^{-\xi^2} e^{-\frac{\xi^2}{p_1^2}} d\xi \int_0^\infty \rho^2 e^{-\frac{\xi^2}{p_2^2}} d\rho \text{ therefore it is enough to prove absolute and uniform convergence of its first term. Substitution $y = 2\zeta + q\sqrt{t}$ reduces it to the following form:}
$$

$$
\frac{1}{2} \int_0^\infty \int_{-\infty}^\infty \rho^2 T_{yo}^2 u_0(x_0 - y\sqrt{t} - qt, y_0) \left[ \int_0^\infty e^{-z^2 + at} \left( \cos \frac{b\zeta}{\sqrt{t}} - 1 \right) \times \right.
$$

$$
\times \cos \left(yz - q\sqrt{t}z + at \sin \frac{h\zeta}{\sqrt{t}}\right) dz \int_0^\infty \xi^k e^{-\xi^2 + bt} \left[ j_\nu\left(\frac{\xi}{\sqrt{t}}\right) - 1\right] j_\nu(2\xi \rho) d\xi d\rho.
$$

It is known from [9] that under the assumptions of Th. 5

$$
\left| \int_0^\infty e^{-z^2 + at} \left( \cos \frac{b\zeta}{\sqrt{t}} - 1 \right) \cos \left(yz - q\sqrt{t}z + at \sin \frac{h\zeta}{\sqrt{t}}\right) dz \right| \leq \frac{M}{1 + y^2}
$$

for $t \geq 1, y > 0$, where $M$ is a positive constant.

Further, it is known from [10] that

$$
\int_0^\infty \xi^k e^{-\xi^2 + bt} \left[ j_\nu\left(\frac{\xi}{\sqrt{t}}\right) - 1\right] j_\nu(2\xi \rho) d\xi
$$

is a finite sum of terms of the following kind:

(26) $\frac{1}{\rho^{2\nu + 2m + t}} \int_0^\infty \xi^{2\nu + m}(2\rho \xi)e^{-\xi^2 + bt} \left[ j_\nu\left(\frac{\xi}{\sqrt{t}}\right) - 1\right] j_{\nu + t + 1}\left(\frac{\xi}{\sqrt{t}}\right) f_{t}(\xi, t) d\xi.$
Here natural \( l \) does not exceed \( m - 1 \), \( f_l \) is bounded, \( j_\nu(z) = z^\nu J_\nu(z) \).

For \( t \geq 1 \) the modulus of expression (26) does not exceed

\[
(27) \quad \frac{\text{const}}{\rho^{2\nu+2m}} \int_0^\infty \xi j_{\nu+m}(2\rho \xi) e^{-\frac{\xi^2+bt}{2\rho}} d\xi.
\]

\[
\xi j_{\nu+m}(2\rho \xi) = \frac{1}{2\rho} (2\rho \xi)^{\nu+m+1} j_{\nu+m}(2\rho \xi) = \frac{(2\rho \xi)^{\nu+m+\frac{1}{2}}}{2\rho} \sqrt{2\rho \xi} j_{\nu+m}(2\rho \xi).
\]

On the other hand, if the conditions of Th. 5 hold (i.e. if \( \frac{b g^2}{4\nu + 4} > 0 \)) then the power of the exponent in (27) does not exceed \(- \alpha \xi^2\) with a positive \( \alpha \) (see prove of L. 10). Then, taking into account the boundedness of function \( \sqrt{t} J_\nu(\tau) \), we obtain that the modulus of expression (27) does not exceed

\[
\frac{\text{const}}{\rho^{2\nu+2m+1-\nu-m-\frac{3}{2}}} \int_0^\infty \xi^{\nu+m+\frac{3}{2}} e^{-\alpha \xi^2} d\xi = \frac{\text{const}}{\rho^{m+\nu+\frac{3}{2}}}.
\]

Thus, taking natural \( m \) from \( (\nu + \frac{3}{2}, \infty) \), we obtain that that there exists \( \beta > 1 \) such that

\[
\int_0^\infty \xi^k e^{-\xi^2 + bt} j_\nu(2\xi \rho) d\xi \leq \frac{\text{const}}{\rho^\beta}.
\]

This estimate is used if \( \rho \geq 1 \). In case of \( \rho \in (0, 1) \) we use the boundedness of the latter integral (as a function of \( t \in [1, \infty) \)), following from the boundedness of \( j_\nu(\cdot) \) and the above-mentioned estimate of the power of the exponent in the integrand, found in L. 1. By the virtue of the boundedness of function \( u_0 \) it completes the proof of absolute and uniform convergence of the first term of integral (25).

Now let us decompose (25) into the following sum:

\[
\int_{\{\xi|<\delta, 0<\rho<\delta\}} + \int_{\mathbb{R}_+^2 \setminus \{\xi|<\delta, 0<\rho<\delta\}} \overset{\text{def}}{=} I_3 + I_4.
\]

By the virtue of its absolute and uniform convergence for any positive \( \varepsilon \) one can choose a positive \( \delta \) such that \( |I_4| \leq \varepsilon \) for any \( t \) from \( [1, \infty) \). Let us fix the chosen \( \delta \) and consider \( I_3 \).
By the virtue of the boundedness of function $u_0$

$$|I_3| \leq \text{const} \int_0^\delta \int_0^\delta e^{-\int_0^\delta \int_0^\delta t - at} \left( \cos \frac{h\zeta}{\sqrt{t}} \right) \times$$

$$\times \cos \left( 2z\zeta + at \sin \frac{hz}{\sqrt{t}} \right) dz \int_0^\infty \xi^k e^{-\xi^2 + bt} [j_\nu(\xi^k t^{-1})] j_\nu(2\xi \rho) d\xi -$$

$$- \frac{\sqrt{\pi}}{2p_1^2} e^{-\frac{(2\xi + \sqrt{t})^2}{4p_1^2}} \Gamma(\nu + 1) \frac{\xi^k e^{-\xi^2}}{r_2^2} d\zeta d\rho.$$ 

By the virtue of L. 4 of paper [9]

$$\int_0^r e^{-\zeta^2 + at} \left( \cos \frac{h\zeta}{\sqrt{t}} \right) \cos \left( 2z\zeta + at \sin \frac{hz}{\sqrt{t}} \right) dz - \frac{\sqrt{\pi}}{2p_1^2} e^{-\frac{(2\xi + \sqrt{t})^2}{4p_1^2}} \xi^k t^{-1} \to 0$$

uniformly with respect to $\zeta \in \mathbb{R}^1$.

This and L. 10 of the present paper imply that there exists $t_0 > 0$ such that for any $t \geq t_0$ the expression inside modulus brackets in (28) does not exceed $\frac{\epsilon}{2} \left( \int_0^\delta \int_0^\delta \rho^k d\zeta d\rho \right)^{-1}$.

The proof of Th. 5 is completed. \(\square\)

Similarly to the regular case [9], if we additionally require the elliptic operator contained at the considered equation to be symmetric, then besides weighted closeness of solutions (statement of Th. 5) weighted stabilization of solution $u(x, y, t)$ takes place either. More exactly, the following assertion is valid:

**Corollary 1.** Let the conditions of Th. 5 are satisfied while operator $A$ is symmetric. Then for any real $l$

$$\lim_{t \to \infty} e^{-t \left( \sum_{i=1}^m a_{iv_0} + \sum_{j=1}^n b_{jr} \right)} u(x, y, t) = l \text{ for any } (x, y) \in \mathbb{R}^{n+m}$$

if and only if

$$\lim_{t \to \infty} \frac{C_{m,n,k}}{t^{m+n+|k|}} \int_{B^+(p,t)} \prod_{i=1}^n y_i^k u_0(x, y) dx dy = l.$$ 

where $B^+(p,t) = \{(x,y) \in \mathbb{R}^{m+n} \mid \sum_{i=1}^m x_i^2/p_i^2 + \sum_{i=1}^n y_i^2/p_{m+1}^2 < t^2\}$,

$$C_{m,n,k} = \frac{\pi^{\frac{m}{2}}}{2^{n-1}(m+n+|k|) \prod_{i=1}^m p_i \prod_{i=1}^{n+1} p_{m+i}^{k+1}}.$$
To prove that it is enough to realize that \( q_1 = \ldots = q_m = 0 \) under the assumptions of Corollary 1, and to apply theorems on stabilizations of singular differential parabolic equations (see e.g. [12]).

**Remark 2.** Since \( T^h_y f(y) = T^{-h}_y f(y) \) then the singular part of operator \( A \) is always symmetric. Then the condition of symmetry of operator \( A \) may be replaced with the condition of symmetry of the following differential-difference operator:

\[
A_{\text{reg}} u \overset{\text{def}}{=} \sum_{i=1}^{m} \left[ \frac{\partial^2 u}{\partial x_i^2} + \sum_{s=1}^{m_i} a_{is} u(x + h_{is}, y, t) \right].
\]

**Remark 3.** If the conditions of Corollary 1 are satisfied then the requirement of symmetry of operator \( A_{\text{reg}} \) can be weakened: it is enough to assume that \( a_i \perp h_i \) for any \( i \in \overline{1, m} \), where vectors \( a_i, h_i \) are defined on the following way: \( a_i = (a_{i1}, \ldots, a_{im_i}) \), \( h_i = (h_{i1}, \ldots, h_{im_i}) \).

**Remark 4.** It is clear from Corollary 1, that in functional-differential case the domains of averaging of the initial-value function (contained at the stabilization condition) are not bounded by segments of spheres anymore: now the bounding surfaces become ellipsoids. Let us recall that in the classical case of differential equations we have the same effect when we replace operator \( \Delta_B \overset{\text{def}}{=} \Delta_x + \sum_{l=1}^{n} B_{k_l, y_l} \) with the following operator with different coefficients at different second derivatives:

\[
\sum_{i=1}^{n} p_i^2 \frac{\partial^2}{\partial x_i^2} + \sum_{l=1}^{n} p_{m+l}^2 B_{k_l, y_l}.
\]

**Remark 5.** Exponential weight arising in the proved theorems on closeness (stabilization respectively) of solutions is not caused by the arising of the non-local terms of the equation. The real reason is the dissipativity of the problem. The specified weight takes place in the classical case too: if all the coefficients \( h_{is} \) and \( g_{lr} \) vanish then limit relation (23) becomes the identity (i.e. holds for all \( t \)). This happens because adding of low-order (more exactly - zero-order) terms to a parabolic equation generally leads its solution out of the class of bounded functions (even in the case of bounded initial-value functions), but its multiplying by the corresponding exponential (with respect to \( t \)) weight returns the solution to the specified class.

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