ON PERSISTENCE OF A DELAY DIFFERENTIAL EQUATION WITH POSITIVE AND NEGATIVE COEFFICIENTS

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Abstract. In this paper we consider the equation

\[ \dot{x}(t) = c(t)x(t) - a(t)x(g(t)), \ t \geq 0, \]

where \( c(t) \geq a(t) \geq 0, \ g(t) \leq t. \) We prove that under some conditions every positive solution of this equation has the following property

\[ 0 < \liminf_{t \to \infty} x(t) < \limsup_{t \to \infty} x(t) < \infty. \]

Impulsive and nonlinear equations with positive and negative coefficients

\[ \dot{x}(t) = c(t)x(t) - f(t, x(g(t))), \ x(\tau_j^+) = I_j(x(\tau_j)), \ \lim_{j \to \infty} \tau_j = \infty, \]

are also considered.

Dedicated to Anatolii Dmitrievich Myshkis on the occasion of his jubilee

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1. Introduction. One of the first monographs on delay differential equations [1] was written in 1951 by A.D. Myshkis. This book had a significant influence on all further development of the theory of functional differential equations. Together with other classes of equations, in [1] and the next monograph [2] A.D. Myshkis considered a class of so-called unstable type linear functional differential equations. In particular, this class includes a linear differential equation with several delays and positive coefficients:

\[
\dot{x}(t) = \sum_{k=1}^{m} a_k(t)x(g_k(t)),
\]

where \(a_k(t) \geq 0\), \(g_k(t) \leq t\).

One of the main results obtained in [1, 2] for equation (1) is the following. If an initial function is positive on some interval, then for the solution \(x(t)\) of the equation we have:

\[
N = \liminf_{t \to \infty} x(t) > 0.
\]

The property (2) (together with the boundedness of solution \(x(t)\)) is called persistence; it is one of important properties for differential equations of Mathematical Biology. For instance, for equations of population dynamics it means guaranteed non-extinction of the population; moreover, the size of the population does not fall beyond the lower bound \(N\).

If we assume in Eq. (1) \(a_i(t) \leq 0\) (a stable type equation), then [3] any positive solution under rather natural constrains tends to zero: \(\lim_{t \to \infty} x(t) = 0\), which means that the zero solution is an attractor for all solutions of the equation.

We consider here the case when Eq. (1) contains both negative and positive coefficients, where the positive term "prevails" over the negative one. Naturally this extends a class of unstable type equations which was considered by A.D. Myshkis. In this paper we obtain persistence conditions for linear differential equations of unstable type with positive and negative coefficients.

Stable type equations with positive and negative coefficients were studied in [4, 5].

We also consider some nonlinear delay differential equations and delay differential equations with impulses.

It is to be noted that it was A.D. Myshkis and V.D. Milman who first introduced the notion of an impulsive differential equation in 1960 [6]. Since then several monographs and more than 1000 papers have been published on impulsive equations.
2. Preliminaries. Consider the scalar linear delay differential equation

\[ \dot{x}(t) + \sum_{i=1}^{n} c_i(t)x(g_i(t)) = f(t), \quad t \geq 0, \]

with the initial condition

\[ x(t) = \varphi(t), \quad t < 0, \quad x(0) = x_0, \]

under the following assumptions

(a1) \( c_i(t) \) is a locally essentially bounded on \([0, \infty)\) function;

(a2) \( g_i(t) \) is a Lebesgue measurable function, \( g_i(t) \leq t \),

\[ \limsup_{t \to \infty} g_i(t) = \infty; \]

(a3) \( \varphi : (-\infty, 0) \to R \) is a Borel measurable bounded function.

**Definition 1.** A solution \( X(t, s) \) of the problem

\[ \dot{x}(t) + \sum_{i=1}^{n} c_i(t)x(g_i(t)) = 0, \quad t \geq s, \]

\[ x(t) = 0, \quad t < s, \quad x(s) = 1, \]

is called a fundamental function of (3).

**Definition 2.** We say that a function is nonoscillatory if it is either eventually positive or eventually negative.

We say that a function \( x(t) \) is persistent if

\[ 0 < \liminf_{t \to \infty} x(t) < \limsup_{t \to \infty} x(t) < \infty. \]

**Lemma 1.** (see [7]) Suppose for (3) conditions (a1)-(a3) hold. Then for the solution of (3), (4) we have the following representation

\[ x(t) = X(t, 0)x_0 - \int_{0}^{t} X(t, s) \sum_{i=1}^{n} c_i(s)\varphi(g_i(s))ds + \int_{0}^{t} X(t, s)f(s)ds, \]

where \( \varphi(t) = 0, \quad t \geq 0. \)

**Lemma 2.** (see [3]) Suppose (a1)-(a3) hold, \( c_i(t) \geq 0 \). If there exists a nonnegative solution of the inequality

\[ u(t) \geq \sum_{i=1}^{n} c_i(t)\exp \left\{ \int_{g_i(t)}^{t} u(s)ds \right\}, \quad t \geq 0; \quad u(t) = 0, \quad t < 0, \]
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and

\[ 0 \leq \varphi(t) \leq x_0, \]

then the solution of initial value problem (3), (4), with \( f(t) \equiv 0 \), is positive.

Consider also the following linear delay equation with positive and negative coefficients

\[ \dot{x}(t) + a(t)x(g(t)) - c(t)x(t) = 0, \quad t \geq 0, \]

and the corresponding linear inequality

\[ \dot{y}(t) + a(t)y(g(t)) - c(t)x(t) \leq 0, \quad t \geq 0. \]

**Lemma 3.** *(see [4, 5])** Suppose \( (a1)-(a3) \) hold,

\[ a(t) \geq c(t) \geq 0, \int_0^\infty [a(s) - c(s)]ds = \infty, \]

and

\[ \limsup_{t \to \infty} c(t)[t - g(t)] < 1. \]

Then

1) If \( y(t) \) is a positive solution of (10) for \( t \geq t_0 \geq 0 \), then

\( y(t) \leq x(t), \quad t \geq t_0 \geq 0 \), where \( x(t) \) is a solution of (9) and \( x(t) = y(t), \quad t \leq t_0 \).

2) For every nonoscillatory solution of (9) we have \( \lim_{t \to \infty} x(t) = 0 \).

3) There exists a positive solution of (9).

Consider now the following linear equation with one delay

\[ \dot{x}(t) = a(t)[x(t) - x(g(t))], \quad t \geq 0. \]

**Lemma 4.** *(8)** Suppose for parameters of Eq. (13) hypotheses \( (a1)-(a3) \) hold, \( g(t) \) is an increasing function,

\[ a(t) \geq 0, \quad \limsup_{t \to \infty} (t - g(t)) < \infty, \quad \limsup_{t \to \infty} \int_{g(t)}^t a(s)ds < 1. \]

Then every solution of (13) has a finite limit. In particular, for the fundamental function \( X(t,s) \) of (13) we have

\[ \sup_{t \geq s \geq 0} |X(t,s)| < \infty. \]
3. **Linear Equations.** In this section we consider the following equation with one delay

\[ x(t) = c(t)x(t) - a(t)x(g(t)), \]

and a corresponding differential inequality

\[ y(t) \geq c(t)y(t) - a(t)y(g(t)), \]

where for parameters of (14) conditions (a1)-(a3) hold.

**Theorem 1.** 1) Suppose (a1)-(a3) hold, \( c(t) \geq a(t) \geq 0 \). Then there exists an eventually positive solution of Eq. (14) and for any positive solution \( x(t) \) of (14) and for the solution \( y(t) \) of (15), with \( y(t) = x(t), t \leq 0 \), we have \( y(t) \geq x(t) > 0 \). Suppose for initial conditions (4) inequality (8) holds. Then for the solution of (14), (4) we have \( x(t) \geq x_0 > 0 \). In particular,

\[ \liminf_{t \to \infty} x(t) \geq x(0) > 0. \]

2) Suppose conditions (11), (12) hold. Then (14) has no persistent solutions (every positive solution tends to zero).

**Proof.** 1) In the space \( L_\infty[0, T] \) of all essentially bounded on \([0, T]\) functions with a usual sup-norm consider the following operator equation

\[ w(t) = c(t) - a(t) \exp \left\{ - \int_{g(t)}^{t} w(s)ds \right\}, \quad t \geq 0; \quad w(t) = 0, t < 0. \]

Denote the sequence: \( w_1(t) = c(t) - a(t), \)

\[ w_n(t) = c(t) - a(t) \exp \left\{ - \int_{g(t)}^{t} w_{n-1}(s)ds \right\}, \quad w_n(t) = 0, t < 0. \]

Inequality \( w_0(t) \geq 0 \) implies \( w_1(t) \geq w_0(t) \). By induction we can prove \( w_n(t) \geq w_{n-1}(t) \geq w_0(t) = c(t) - a(t) \geq 0 \) and \( w_n(t) \leq c(t) \).

There exists a pointwise limit of the nondecreasing nonnegative sequence \( w_n(t) \). Let \( w(t) = \lim_{n \to \infty} w_n(t) \), then by the Lebesgue Convergence Theorem \( w(t) \) is locally integrable and

\[ \lim_{n \to \infty} (Fw_n)(t) = (Fw)(t), \]

where operator \( F \) is denoted by the right-hand side of (17). Thus (18) implies that \( w \) is a nonnegative solution of Eq. (17).
Hence function $x$ defined by the equality
\begin{equation}
(19) \quad x(t) = e^{\int_0^t w(s) \, ds}, \quad x(t) = 0, \, t < 0,
\end{equation}
is a positive solution of equation (14). Since $x(0) = 1$ then $x(t) = X(t, 0)$, where $X(t, s)$ is the fundamental function of Eq. (14). We have $X(t, 0) > 0$. Similarly we can prove that $X(t, s) > 0, \, t \geq s \geq 0$.

For a solution $y(t)$ of inequality (15) we have
\begin{equation}
\dot{y}(t) = c(t)y(t) - a(t)y(g(t)) + f(t),
\end{equation}
where $f(t) \geq 0$. Then by Lemma 1
\begin{equation}
y(t) = x(t) + \int_0^t X(t, s)f(s) \, ds,
\end{equation}
where $x(t)$ is a solution of (14) with $x(t) = y(t), \, t \leq 0$. If $x(t) > 0$, then $y(t) \geq x(t) > 0$.

Suppose now that $x(t)$ is a solution of (14) and for initial conditions inequality (8) holds. By substituting
\begin{equation}
x(t) = \exp \left\{ \int_0^t c(s) \, ds \right\} z(t), \quad x(t) = z(t), \quad t \leq 0,
\end{equation}
into Eq. (14) we obtain that $z(t)$ is a solution of the linear equation with a nonnegative coefficient
\begin{equation}
\dot{z}(t) + a(t) \exp \left\{ - \int_{g(t)}^t c(s) \, ds \right\} z(g(t)) = 0,
\end{equation}
and with the same initial conditions (4) for which inequality (8) holds. Inequality (7) for this equation has the form
\begin{equation}
u(t) \geq a(t) \exp \left\{ - \int_{g(t)}^t c(s) \, ds \right\} \exp \left\{ \int_{g(t)}^t u(s) \, ds \right\}; \quad u(t) = 0, \, t \leq 0.
\end{equation}
A nonnegative function $u(t) = a(t)$ is a solution of this inequality. Lemma 2 implies $z(t) > 0$. Hence also $x(t) > 0$. Similarly to the above proof (which is given for the fundamental function $X(t, s)$), $x(t)$ can be presented in the form
\begin{equation}
x(t) = x(0) \exp \left\{ \int_0^t w(s) \, ds \right\}, \quad t > 0,
\end{equation}
where \( w(t) \) is a nonnegative solution of Eq. (17). Hence \( x(t) \geq x(0), t > 0 \), and inequality (16) holds, which completes the proof.

The statement of 2) follows from Lemma 3. \( \Box \)

Let us demonstrate the sharpness of conditions of Theorem 1.

**Example 1.** If the condition \( c(t) \geq a(t) \) does not hold, then a solution with initial conditions satisfying (8) can become negative. Let \( \varphi(t) \equiv 1, x_0 = 1.1, g(t) = t - 1, a(t) = 2, c(t) = 1 \). Then the solution of the equation

\[
\dot{x}(t) = x(t) - 2x(t - 1) \equiv x(t) - 2, \quad 0 \leq t \leq 1.
\]

is \( x(t) = -0.9e^t + 2 \) for \( t \in [0, 1] \), so \( x(1) = -0.9e + 2 < 0 \). Hence the solution becomes negative at \( x = \ln(2/0.9) \approx 0.7985 \).

**Example 2.** Similarly, if \( c(t) \geq a(t) \) holds but (8) is not satisfied, then the solution can become negative. For example, let \( \varphi(t) \equiv 10, x_0 = 1, g(t) = t - 1, a(t) = 0.5, c(t) = 1 \). Then the solution of the equation

\[
\dot{x}(t) = x(t) - 0.5x(t - 1) = x(t) - 5, \quad 0 \leq t \leq 1,
\]

with \( x(0) = 1 \), is \( x(t) = 5 - 4e^t \), so \( x(1) = 5 - 4e < 0 \) and the solution becomes negative at \( x = \ln(1.25) \approx 0.223 \).

In addition to the positiveness of solutions Theorem 1 claims that as far as (8) is satisfied then the solution does not tend to zero (moreover, it is not less than the initial value). However if (8) does not hold then a positive solution can become less than the initial value and can tend to zero as Example 3 demonstrates.

**Example 3.** The equation \( \dot{x}(t) = x(t) - \frac{t^2 - 1}{t^2} x(t - 1) \) has a solution \( x = \frac{1}{t} \) as can be easily checked. We can begin anywhere at \( t > 1 \) including the previous part of the solution as prehistory. The solution tends to zero; (16) does not hold since (8) is not satisfied for any initial point.

**Theorem 2.** Suppose \( (a1)-(a3) \) hold,

\[
(20) \quad g(t) \text{ is an increasing function, } \limsup_{t \to \infty} (t - g(t)) < \infty,
\]

\[
(21) \quad a(t) \geq 0, \quad \limsup_{t \to \infty} \int_{g(t)}^{t} a(s)ds < 1, \quad \int_{0}^{\infty} |c(s) - a(s)|ds < \infty.
\]

Then every solution of (14) is bounded.

**Proof.** Rewrite Eq. (14) in the form:

\[
\dot{x}(t) = a(t)(x(t) - x(g(t))) + [c(t) - a(t)]x(t).
\]
Denote by $Y(t, s)$ the fundamental function of the following equation
\[ \dot{y}(t) = a(t)[y(t) - y(g(t))]. \]

Lemma 1 implies that for the solution of (14), (4) we have
\[ x(t) = Y(t, 0)x(0) + \int_0^t Y(t, s)[c(s) - a(s)]x(s)ds. \]

Conditions (20), (21) and Lemma 4 imply
\[ C = \sup_{t \geq s \geq 0} |Y(t, s)| < \infty. \]

Hence
\[ |x(t)| \leq C|x(0)| + C \int_0^t |c(s) - a(s)||x(s)|ds. \]

Then
\[ |x(t)| \leq C|x(0)| \exp \left\{ C \int_0^t |c(s) - a(s)|ds \right\} \leq C|x(0)| \exp \left\{ C \int_0^\infty |c(s) - a(s)|ds \right\} < \infty. \]

We have $\sup_{t \geq 0} |x(t)| < \infty$, which completes the proof. □

**Remark 1.** It is well known that if
\[ \int_0^\infty |a(s)|ds < \infty, \quad \int_0^\infty |c(s)|ds < \infty, \]
then every solution of (14) has a finite limit.

**Corollary 1.** Suppose the conditions of Theorem 2 hold and $c(t) \geq a(t) \geq 0$. Then every solution of (14), (4) with initial conditions satisfying (8), is persistent.

**4. Impulsive and Nonlinear Equations.** Now let us assume that the equation (14) is subject to linear impulsive perturbations
\[ x(\tau_j^+) = b_jx(\tau_j), \quad j = 1, 2, \ldots, \]
and the following conditions hold:
(a4) $t_0 < \tau_1 < \tau_2 < \ldots < \tau_k < \ldots$ satisfy $\lim_{j \to \infty} \tau_j = \infty$;
(a5) $b_j > 0, \quad j = 1, 2, \ldots$. 


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If we make a substitution \([9, 10, 11, 12]\)

\[(23)\]

\[z(t) = \prod_{0 \leq \tau_j \leq t} b_j^{-1} x(t),\]

then the function \(z(t)\) is an absolutely continuous solution of the equation

\[(24)\]

\[\dot{z}(t) = c(t) z(t) - a_1(t) z(g(t)),\]

where

\[(25)\]

\[a_1(t) = a(t) \prod_{g(t) \leq \tau_j \leq t} b_j^{-1}.\]

We also consider the following impulsive inequalities

\[(26)\]

\[x(\tau_j^+) \leq b_j x(\tau_j), \quad j = 1, 2, \ldots ,\]

and

\[(27)\]

\[x(\tau_j^+) \geq b_j x(\tau_j), \quad j = 1, 2, \ldots ,\]

Thus, Theorem 1 and Theorem 2 applied to the continuous solution \(z(t)\), as well as the representation

\[x(t) = \prod_{0 \leq \tau_j \leq t} b_j z(t),\]

imply the following results.

**THEOREM 3.** Suppose \((a1)-(a5)\) hold,

\[c(t) \geq a_1(t), \quad \liminf_{n \to \infty} \prod_{j=1}^{n} b_j > 0.\]

Then there exists an eventually positive solution of \((14), (22)\) and for any positive solution \(x(t)\) of \((14),(22)\) and for the solution \(y(t)\) of \((15),(27)\), with \(y(t) = x(t), \ t \leq 0,\) we have \(y(t) \geq x(t) > 0.\)

Suppose for initial conditions (4) inequality (8) holds. Then for each solution of \((14), (4),(22)\) we have

\[\liminf_{t \to \infty} x(t) > 0.\]

Suppose conditions \((11), (12)\) hold, where \(a(t)\) is replaced by \(a_1(t),\) and

\[\limsup_{n \to \infty} \prod_{j=1}^{n} b_j < \infty.\]

Then \((14), (22)\) has no persistent solutions (every positive solution tends to zero).
THEOREM 4. Suppose (a1)-(a5), (20) and (21) hold, where \( a(t) \) is replaced by \( a_1(t) \), and \( \limsup_{n \to \infty} \prod_{j=1}^{n} b_j < \infty \). Then every solution of impulsive equation (14), (22) is bounded.

COROLLARY 2. Suppose (a1)-(a5), (20) and (21) hold, where \( a(t) \) is replaced by \( a_1(t) \),

\[
\begin{align*}
    c(t) &\geq a_1(t), \\
    \liminf_{n \to \infty} \prod_{j=1}^{n} b_j &> 0, \\
    \limsup_{n \to \infty} \prod_{j=1}^{n} b_j &< \infty.
\end{align*}
\]

Then every solution of impulsive equation (14), (22), (4), with initial conditions satisfying (8), is persistent.

Further, let us assume that the negative term in (14) is nonlinear

\[
(28) \quad \dot{x}(t) = c(t)x(t) - f(t, x(g(t))),
\]

where the following condition holds:

(a6) \( f(t, u) \) satisfies Caratheodory conditions: it is Lebesgue measurable in the first argument and continuous in the second one.

We will assume that the initial value problem (28)-(4) has a unique global solution \( x(t), t \geq 0 \).

THEOREM 5. Suppose (a1)-(a3), (a6) hold.

1) If

\[
(29) \quad f(t, u) \geq a(t)u, \quad u > 0,
\]

and (11), (12) are satisfied then (28) has no persistent solutions (every positive solution tends to zero).

2) If

\[
(30) \quad f(t, u) \leq c(t)u, \quad u > 0,
\]

and \( c(t) \geq 0 \) then any solution of (28), (4), with the initial conditions satisfying (8), has property (16).

Proof. 1) Due to (29) the solution of (28) is also a solution of inequality (10). Thus by Lemma 3 any positive solution tends to zero.

2) Similarly, under (30) the solution of (28) is also a solution of inequality (15), with \( a(t) = c(t) \). Thus by Theorem 1 every solution with the initial conditions satisfying (8) has property (16). \( \square \)
Example 4. The equation

$$\dot{x}(t) = x(t) - \frac{2x(t-0.2)(x(t-0.2) + 1)}{1 + \sqrt{x(t-0.2)}}$$

has no persistent solutions. As can be easily checked,

$$\frac{2u(u + 1)}{1 + \sqrt{u}} \geq 1.6u \geq u, \quad u \geq 0,$$

thus hypotheses of Theorem 5, part 1), are satisfied and any positive solution of this equation tends to zero.

Example 5. Any solution of the equation

$$\dot{x}(t) = x(t) - \frac{x(t-2)}{1 + 0.5x^2(t-2)}$$

with the initial conditions satisfying (8) has property (16), since the the hypotheses of Theorem 5, part 2), are satisfied due to the inequality

$$\frac{u}{1 + u^2} \leq u, \quad u \geq 0.$$

Theorem 6. Suppose (a1)-(a3), (a6), (20), (21) and (29) hold. Then every positive solution of (28) is bounded.

Proof. Suppose $$x(t)$$ is a solution of (28). Inequality (29) implies

$$\dot{x}(t) = c(t)x(t) - a(t)x(g(t)) - h(t),$$

where $$h(t) \geq 0$$. This equation can be rewritten in the form

$$\dot{x}(t) = a(t)(x(t) - x(g(t))) + [c(t) - a(t)]x(t) - h(t).$$

Denote by $$Y(t, s)$$ the fundamental function of the following equation

$$\dot{y}(t) = a(t)[y(t) - y(g(t))].$$

By Lemma 1 for the solution of (14), (4) we have

$$x(t) = Y(t, 0)x(0) + \int_0^t Y(t, s)[c(s) - a(s)]x(s)ds - \int_0^t Y(t, s)h(s)ds.$$ 

The proof of Theorem 1 implies $$Y(t, s) > 0$$. Thus

$$0 < x(t) \leq Y(t, 0)x(0) + \int_0^t Y(t, s)[c(s) - a(s)]x(s)ds.$$
Similar to the proof of Theorem 2 we have
\[ 0 < x(t) \leq C x(0) \exp \left\{ C \int_0^t |c(s) - a(s)| \, ds \right\} \]
\[ \leq C x(0) \exp \left\{ C \int_0^\infty |c(s) - a(s)| \, ds \right\} < \infty, \]
where
\[ C = \sup_{t,s \geq 0} Y(t,s) < \infty. \]

Finally, \( \sup_{t \geq 0} x(t) < \infty \), which completes the proof. \( \square \)

**COROLLARY 3.** Suppose \((a1)-(a3), (a6), (20)\) and \( (21) \) hold,
\[ 0 \leq a(t)u \leq f(t,u) \leq c(t)u, \quad u > 0. \]
Then every solution of \((28), (4)\), with initial conditions satisfying \((8)\), is persistent.

We also introduce nonlinear impulsive conditions for \((28)\)
\[ (31) \quad x(\tau_j^+) = I_j(x(\tau_j)), \quad j = 1, 2, \ldots, \]
and will assume that the initial value problem \((28), (31), (4)\) has a unique global solution \(x(t), \quad t \geq 0.\)

**THEOREM 7.** 1) Suppose there exist such \(a(t), b_j\) that \((a1)-(a6)\) hold and
\[ (32) \quad f(t,u) \geq a(t)u, \quad I_j(u) \geq b_j u, \quad u \geq 0. \]
If \((11)\) and \((12)\) hold, where \(a(t)\) is replaced by \(a_1(t)\) (which is denoted by \(25)\), \( \limsup_{n \to \infty} \prod_{j=1}^n b_j < \infty \), then \((28), (31)\) has no persistent solutions (every positive solution tends to zero).
2) Suppose there exists a sequence \(b_j\), such that \((a1)-(a6)\) hold,
\[ c(t) \geq a_1(t) \geq 0, \quad \liminf_{n \to \infty} \prod_{j=1}^n b_j > 0 \text{ and} \]
\[ (33) \quad f(t,u) \leq c(t)u, \quad I_j(u) \leq b_j u, \quad u \geq 0. \]
Then for any solution of \((28), (31), (4)\), with the initial conditions satisfying \((8)\), we have \( \liminf_{t \to \infty} x(t) > 0. \)
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