A FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATION IN BANACH ALGEBRAS

B. C. DHAGE *

Abstract. In this paper, some existence theorems for functional nonlinear first order integro-differential equations are proved in a Banach algebra, using a nonlinear alternative of Leray-Schauder type. The existence of the extremal positive solutions is also proved under certain monotonicity condition.

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1. Introduction. Given a closed and bounded interval \( J = [0, 1] \) in \( \mathbb{R} \), the set of all real numbers, consider the following nonlinear functional integro-differential equation (in short FIDE)

\[
\begin{aligned}
\left( \frac{x(t)}{f(t, x(\theta(t)))} \right)' &= g(t, x(\mu(t)), \int_0^{\sigma(t)} k(t, s, x(\eta(s))) \, ds), \quad \text{a.e. } t \in J \\
x(0) &= x_0 \in \mathbb{R}
\end{aligned}
\]

where \( f : J \times \mathbb{R} \to \mathbb{R} - \{0\} \) is continuous, \( g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) and \( \theta, \mu, \sigma, \eta : J \to J \) are continuous with \( \theta(0) = 0 \).

By a solution of the FIDE (1) we mean a function \( x \in AC(J, \mathbb{R}) \) that satisfies the equations in (1), where \( AC(J, \mathbb{R}) \) is the space of all absolutely continuous real-valued functions on \( J \).

The FIDE (1) is new to the literature and so the results of this paper are new to the theory of nonlinear differential equations. The special cases of the FIDE (1) have already been studied in the literature by several authors for various aspects of the solution. For example, if \( f(t, x) = 1, \forall (t, x) \in J \times \mathbb{R} \)

* Kasubai, Gurukul Colony, Ahmedpur-413 515, Dist: Latur, Maharashtra, India
and \( \mu(t) = \sigma(t) = \eta(t) = t, \ \forall \ t \in J \), then the FIDE (1) reduces to the integro-differential

\[
\begin{align*}
\dot{x}'(t) &= g(t, x(t), \int_0^t k(t, s, x(s)) \, ds), \ \text{a.e. } t \in J \tag{2} \\
x(0) &= x_0.
\end{align*}
\]

There is an abundances literature on the FIDE (2). See Martin [7] and the references therein. Again a special case of the FIDE (1) in the form of

\[
\left( \frac{x(t)}{f(t, x(\theta(t)))} \right)' = g(t, x(\eta(t))), \ \text{a.e. } t \in J \tag{3}
\]

\[
x(0) = x_0 \in \mathbb{R}
\]

has been discussed recently in Dhage et. al. [3] for the existence theorems under mixed Lipschitz and compactness type conditions. The physical situation in which the FIDE (1) occurs are yet to be investigated. But the study of the FIDE (1) is definitely a new vistas in the theory of differential equations. This is a main motivation for the present paper. The rest of the paper is organized as follows. In the following section 2, an existence theorem for the FIDE (1) is proved under certain Lipschitzicity and Carathéodory conditions. Section 3 deals with the existence of extremal solutions and finally an illustrative example is given in section 4.

2. Existence Theory. Let \( B(J, \mathbb{R}) \) denote the space of all bounded real-valued functions on \( J \). By \( AC(J, \mathbb{R}) \) we denote the space of all absolutely continuous real-valued functions on \( J \). Define a norm \( \| \cdot \| \) in \( AC(J, \mathbb{R}) \) by

\[
\| x \| = \max_{t \in J} |x(t)|.
\]

Denote by \( L^1(J, \mathbb{R}) \) the space of all Lebesgue integrable functions on \( J \) with the usual norm \( \| \cdot \|_{L^1} \) given by

\[
\| x \|_{L^1} = \int_0^a |x(t)| \, dt.
\]

The FIDE (1) is equivalent to the functional integral equation (in short FIE)

\[
x(t) = [f(t, x(\theta(t)))) \left( \frac{x_0}{f(0, x_0)} \right)
\]

\[
+ \int_0^t g(s, x(\mu(s)), \int_0^{(\sigma(s)} k(s, \tau, x(\eta(\tau))) \, d\tau) \, ds \right) \, ds, \ t \in J.
\]
We shall apply the following nonlinear alternative of Leray-Schauder type for proving the existence results of FIDE (1).

**Theorem 1.** Let $B_r(0)$ and $\overline{B}_r(0)$ respectively denote an open and closed ball in a Banach algebra $X$ centered at origin of radius $r$, for some real number $r > 0$. Let $A : X \to X$ and $B : \overline{B}_r(0) \to X$ be two operators such that

(a) $A$ is Lipschitzian with a Lipschitz constant $\alpha$,

(b) $B$ is completely continuous and

(c) $\alpha M < 1$, where $M = \|B(\overline{B}_r(0))\| = \sup\{\|Bx\| : x \in \overline{B}_r(0)\}$.

Then either

(i) the equation $Ax \circ Bx = x$ has a solution in $\overline{B}_r(0)$,

or

(ii) there exists an $u \in X$ with $\|u\| = r$ such that $\lambda A \left( \frac{u}{\lambda} \right) Bu = u$ for some $0 < \lambda < 1$.

**Proof.** The proof appears in Dhage et. al. [3], but some discrepancies crept in and so the proof goes wrong. Here we remove these discrepancies and give the correct proof. Let $y \in \overline{B}_r(0)$ and define a mapping $A_y : X \to X$ by

$$A_y x = Ax \circ By, x \in X.$$ 

Notice that $A_y$ is a contraction with a contraction constant $\alpha M < 1$, since

$$\|A_y x_1 - A_y x_2\| \leq \|Ax_1 - Ax_2\||By|| \leq \alpha M \|x_1 - x_2\|$$

whenever $x_1, x_2 \in X$. The Banach contraction principle implies that there is a unique point $x^* \in X$ such that

$$A_y(x^*) = x^*$$

or, equivalently

$$x^* = Ax^* By.$$ 

Define a mapping $N : \overline{B}_r(0) \to X$ by

$$Ny = z$$

where $z \in X$ is the unique solution of the equation

$$z = A(z) By, y \in \overline{B}_r(0).$$
We show that $N$ is continuous. Let \( \{y_n\} \) be a sequence in \( \overline{B}_r(0) \) converging to a point \( y \). Since \( \overline{B}_r(0) \) is closed, \( y \in \overline{B}_r(0) \). Now

\[
\|N y_n - Ny\| = \|AN(y_n)B(y_n) - AN(y)By\| \\
\leq \|AN(y_n)B(y_n) - AN(y)By_n\| + \|AN(y)By_n - AN(y)By\| \\
\leq \|AN(y_n) - AN(y)\||B(y_n)\| + \|AN(y)\||B(y_n) - By\| \\
\leq \alpha M \|N y_n - Ny\| + \|ANy\||B(y_n) - By\|
\]

and hence

\[
\|N y_n - Ny\| \leq \frac{\|ANy\|}{1 - \alpha M} \|B(y_n) - By\|.
\]

This shows that $N$ is continuous on \( \overline{B}_r(0) \). Next we show that $N$ is a compact operator on \( \overline{B}_r(0) \). Now for any \( z \in \overline{B}_r(0) \) we have

\[
\|Az\| \leq \|A0\| + \|Az - A0\| \\
\leq \|A0\| + \alpha \|z - 0\| \\
\leq c
\]

where \( c = \|A0\| + \alpha r \).

Let \( \epsilon > 0 \) be given. Since $B$ is completely continuous, $B(\overline{B}_r(0))$ is totally bounded. Then there is a set \( Y = \{y_1, \ldots, y_n\} \) in \( \overline{B}_r(0) \) such that

\[
B(\overline{B}_r(0)) \subset \bigcup_{i=1}^{n} B_\delta(w_i),
\]

where \( w_i = B(y_i) \) and \( \delta = \left(\frac{1 - (\alpha M)}{c}\right) \epsilon \). Therefore for any \( y \in \overline{B}_r(0) \) we have a \( y_k \in Y \) such that

\[
\|B y - B y_k\| \leq \left(\frac{1 - (\alpha M)}{c}\right) \epsilon.
\]

Also we have

\[
\|N y - Ny_k\| \leq \|Az By - Az_k By_k\| \\
\leq \|Az By - Az_k By\| + \|Az_k By - Az_k Bz_k\| \\
\leq \|Az - Az_k\||B y\| + \|Az_k||B y_k - By\| \\
\leq (\alpha M)\|z - z_k\| + \|Az||B y_k - By\| \\
\leq (\alpha M)\|N y - Ny_k\| + \|Az||B y_k - By\| \\
\leq \frac{c}{1 - (\alpha M)} \|B y - B y_k\| \\
\leq \epsilon.
\]
This is true for every $y \in \overline{B}_r(0)$ and hence

$$N(\overline{B}_r(0)) \subset \bigcup_{i=1}^{n} B_t(w_i).$$

As a result $N(\overline{B}_r(0))$ is totally bounded. Since $N$ is continuous, it is a compact operator on $\overline{B}_r(0)$. Now an application of nonlinear alternative of Leary-Schauder [8] implies that either (i) $x = Nx$ for some $x \in \overline{B}_r(0)$ or (ii) there is an element $u \in \partial \overline{B}_r(0)$ and a real number $\lambda \in (0, 1)$ such that $u = \lambda Nu$.

Assume first that $x \in \overline{B}_r(0)$ is a fixed point operator $N$. Then by the definition of $N$ we obtain

$$x = Nx = A(Nx)Bx = Ax Bx,$$

and so the operator equation $x = Ax Bx$ has a solution in $\overline{B}_r(0)$.

Suppose that there is an element $u \in \partial \overline{B}_r(0)$ and a real number $\lambda \in (0, 1)$ such that $u = \lambda Nu$. Then

$$\frac{u}{\lambda} = Nu = A(Nu)Bu = A \left( \frac{u}{\lambda} \right) Bu,$$

so that

$$u = \lambda A \left( \frac{u}{\lambda} \right) Bu.$$

This completes the proof. 0

We need the following definition in the sequel.

**Definition 1.** A function $\beta : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called Carathéodory if

(i) $t \to \beta(t, x, y)$ is measurable for each $x, y \in \mathbb{R}$, and

(ii) $(x, y) \to \beta(t, x, y)$ is continuous almost everywhere for $t \in J$.

Further a Carathéodory function $\beta(t, x, y)$ is called $L^1$-Carathéodory if

(iii) for each real number $r > 0$, there exists a function $h_r \in L^1(J, \mathbb{R})$ such that

$$|\beta(t, x, y)| \leq h_r(t), \quad \text{a.e. } t \in J$$

for all $x \in \mathbb{R}$ with $|x| \leq r$ and $y \in \mathbb{R}$.

**Definition 2.** A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a $\mathcal{D}$-function if it satisfies

(i) $\psi$ is continuous,
(ii) \( \psi \) is nondecreasing, and
(iii) \( \psi \) is scalarly submultiplicative, that is, \( \psi(\lambda r) \leq \lambda \psi(r) \) for all \( \lambda > 0 \) and \( r \in \mathbb{IR}^+ \).

The class of all \( \mathcal{D} \)-functions on \( \mathbb{IR}^+ \) is denoted by \( \Psi \). There do exist \( \mathcal{D} \)-functions on \( \mathbb{R} \). Actually the function \( \psi : \mathbb{IR}^+ \to \mathbb{IR}^+ \) defined by \( \psi(r) = \ell r, \) \( \ell > 1 \) satisfies all the conditions (i) – (iii) mentioned above and hence a \( \mathcal{D} \)-function on \( \mathbb{IR}^+ \).

We consider the following set of assumptions:
(A_0) The functions \( \theta, \mu, \sigma \) and \( \eta \) are continuous with \( \theta(0) = 0 \).
(A_1) There exists a function \( \gamma \in \mathcal{B}(J, \mathbb{R}) \) with bound \( \|\gamma\| \) such that
\[
|f(t, x) - f(t, y)| \leq \gamma(t)|x - y|, \quad \text{a.e. } t \in J
\]
for all \( x, y \in \mathbb{R} \).
(A_2) The function \( g(t, x, y) \) is \( L^1 \)-Carathéodory.
(A_3) The function \( k : J \times J \times \mathbb{R} \to \mathbb{R} \) is continuous and there exists a function \( \alpha \in L^1(J, \mathbb{IR}^+) \), such that
\[
|k(t, s, y)| \leq \alpha(s)|y| \quad \text{a.e. } t, s \in J \text{ and } y \in \mathbb{R}.
\]
(A_4) There exists a function \( p \in L^1(J, \mathbb{IR}^+) \) and a \( \mathcal{D} \)-function \( \psi \in \Psi \) such that
\[
|g(t, u, v)| \leq p(t)\psi(|u| + |v|) \quad \text{a.e. } t \in J
\]
for each \( u, v \in \mathbb{R} \).

**Theorem 2.** Assume that the hypotheses \((A_0)-(A_4)\) hold. Suppose that there exists a real number \( r > 0 \) such that
\[
(5) \quad r > \frac{F \left( \left| \frac{x_0}{f(0, x_0)} \right| + \|p\|_{L^1} (1 + \|\alpha\|_{L^1}) \psi(r) \right)}{1 - \|\gamma\| \left( \left| \frac{x_0}{f(0, x_0)} \right| + \|p\|_{L^1} (1 + \|\alpha\|_{L^1}) \psi(r) \right)},
\]
where
\[
\|\gamma\| \left( \left| \frac{x_0}{f(0, x_0)} \right| + \|p\|_{L^1} (1 + \|\alpha\|_{L^1}) \psi(r) \right) < 1,
\]
and \( F = \sup_{t \in J} |f(t, 0)| \). Then the FIDE (1) has a solution \( u \) on \( J \) with \( \|u\| \leq r \).

**Proof.** Let \( X = BM(J, \mathbb{R}) \) and consider a closed ball \( \overline{B}_r(0) \) in \( X \) centered at origin and of radius \( r \), where the real number \( r > 0 \) satisfies the inequalities in (5). Define two operators \( A : X \to X \) and \( B : \overline{B}_r(0) \to X \) by
\[
Ax(t) = f(t, x(\theta(t))), \quad t \in J,
\]
and
Now solving FIDE (1) is equivalent to solving FIE (4), which is further equivalent to solving the operator equation

\[(8)\quad Ax(t)Bx(t) = x(t), \quad t \in J.\]

We shall show that the operators \(A\) and \(B\) satisfy all the conditions of Theorem 1. Let \(x_1, x_2 \in X\). Then by (A2),

\[|Ax(t) - Ay(t)| = |f(t, x(\theta(t))) - f(t, y(\theta(t)))| \leq \gamma(t)|x(\theta(t)) - y(\theta(t))| \leq ||\gamma|| ||x - y||.\]

Taking the maximum over \(t\), in the above inequality yields

\[||Ax - Ay|| \leq ||\gamma|| ||x - y||,\]

and so \(A\) is a Lipschitzian with a Lipschitz constant \(||\gamma||\).

Next the hypothesis (A3) together with the Lebesgue dominated convergence theorem implies that the operator \(B : \overline{B}_r(0) \rightarrow X\) is continuous. We shall show that \(B\) is compact. Let \(\{x_n\}\) be a sequence in \(\overline{B}_r(0)\). Then \(||x_n|| \leq r\) for each \(n \in \mathbb{N}\). From (A2) it follows that

\[||Bx_n|| \leq \left|\frac{x_0}{f(0, x_0)}\right| + \int_0^t \left|g(s, x_n(s), \int_0^{\sigma(s)} k(s, \tau, x_n(\eta(\tau))) d\tau)\right| ds \leq \left|\frac{x_0}{f(0, x_0)}\right| + ||h_r||_{L^1},\]

(where \(h_r\) is given as in Definition 1 (iii)). As a result \(\{Bx_n : n \in \mathbb{N}\}\) is a uniformly bounded set in \(X\). Let \(t_1, t_2 \in J\). Notice that

\[|Bx_n(t_1) - Bx_n(t_2)| \leq \int_{t_1}^{t_2} \left|g(s, x_n(\mu(s)), \int_0^{\sigma(s)} k(s, \tau, x(\eta(\tau))) d\tau)\right| ds \leq \int_{t_1}^{t_2} h_r(s) ds \rightarrow 0, \quad \text{as} \quad t_1 \rightarrow t_2.\]

From this we conclude that \(\{Bx_n : n \in \mathbb{N}\}\) is an equi-continuous set in \(X\). Hence \(B : \overline{B}_r(0) \rightarrow X\) is compact by Arzela-Ascoli theorem. Notice for any
$x \in B_r(0),$
\[
M = \|B(B_r(0))\|
= \left| \frac{x_0}{f(0, x_0)} \right|
+ \sup_{x \in B(B_r(0))} \left\{ \sup_{t \in J} \int_{0}^{t} \left| g(s, x(\mu(s))), \int_{0}^{\sigma(s)} k(s, \tau, x(\eta(\tau))) d\tau \right| ds \right\}
\leq \left| \frac{x_0}{f(0, x_0)} \right| + \|p\|_{L^1}(1 + \|\alpha\|_{L^1})\psi(r),
\]
and so from (5)
\[
\|\gamma\| M \leq \|\gamma\| \left( \left| \frac{x_0}{f(0, x_0)} \right| + \|p\|_{L^1}(1 + \|\alpha\|_{L^1})\psi(r) \right) < 1.
\]
Now we are ready to apply Theorem 1. Note that conclusion (ii) cannot hold. To see this, let there be an $u \in X$ with $\|u\| = r$ satisfying $\lambda A \left( \frac{u}{\lambda} \right) Bu = u$ for some $0 < \lambda < 1.$ Then for any $t \in J,$
\[
|u(t)| \leq \left| \lambda \left( f \left( t, \frac{u}{\lambda}(\theta(t)) \right) - f(t, 0) \right) + \lambda |f(t, 0)| \right|
\times \left( \left| \frac{x_0}{f(0, x_0)} \right| + \int_{0}^{t} p(s)\psi \left( \|u(\mu(s)) + \int_{0}^{\theta(s)} \alpha(s)|u(\eta(s))| ds \right) ds \right) ds
\leq \left[ \|u\| + F \right] \left( \left| \frac{x_0}{f(0, x_0)} \right| + \int_{0}^{t} p(s)\psi \left( \|u\| + \int_{0}^{1} \alpha(s)|u\| ds \right) ds \right)
\leq \left( \|u\| + F \right) \left( \left| \frac{x_0}{f(0, x_0)} \right| + \|p\|_{L^1}(1 + \|\alpha\|_{L^1})\psi(\|u\|) \right)
\leq \frac{F \left( \left| \frac{x_0}{f(0, x_0)} \right| + \|p\|_{L^1}(1 + \|\alpha\|_{L^1})\psi(\|u\|) \right)}{1 - \|\gamma\| \left( \left| \frac{x_0}{f(0, x_0)} \right| + \|p\|_{L^1}(1 + \|\alpha\|_{L^1})\psi(\|u\|) \right)}.
\]
Taking the supremum over $t$, we obtain
\[
\|u\| \leq \frac{F \left( \left| \frac{x_0}{f(0, x_0)} \right| + \|p\|_{L^1}(1 + \|\alpha\|_{L^1})\psi(\|u\|) \right)}{1 - \|\gamma\| \left( \left| \frac{x_0}{f(0, x_0)} \right| + \|p\|_{L^1}(1 + \|\alpha\|_{L^1})\psi(\|u\|) \right)}.
\]
Substituting $\|u\| = r$ in the above inequality yield
\[
r \leq \frac{F \left( \left| \frac{x_0}{f(0, x_0)} \right| + \|p\|_{L^1}(1 + \|\alpha\|_{L^1})\psi(r) \right)}{1 - \|\gamma\| \left( \left| \frac{x_0}{f(0, x_0)} \right| + \|p\|_{L^1}(1 + \|\alpha\|_{L^1})\psi(r) \right)}.
\]
This contradicts to (5) and so the conclusion (i) of Theorem 1 holds. As a result the functional FIDE (1) has a solution $x$ on $J$ with $\|x\| \leq r$. This completes the proof. □

REMARK 1. Note that our Theorem 2 does not require the following hypothesis assumed in Dhage and O’Regan [5].

(A4) The function $(t, x) \rightarrow \frac{x}{f(t, x)}$ is well defined and

$$\frac{x}{f(t, x(\theta))} = \frac{x}{f(t, y(\theta))} \quad \text{implies} \quad x = y, \; \forall \; t \in J \quad \text{and for} \quad x, y \in C(J, \mathbb{R}).$$

Also see Dhage and Ntouyas [4].

3. Existence of Extremal Solutions. A non-empty closed set $K$ in a Banach algebra $X$ is called a cone if (i) $K + K \subseteq K$, (ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}$, $\lambda \geq 0$ and (iii) $\{-K\} \cap K = 0$, where 0 is the zero element of $X$. A cone $K$ is called positive if (iv) $K \circ K \subseteq K$, where “$\circ$” is a multiplication composition in $X$. We introduce an order relation $\leq$ in $K$ as follows. Let $x, y \in X$. Then $x \leq y$ if and only if $y - x \in K$. A cone $K$ is called normal if the norm $\| \cdot \|$ is monotone increasing on $K$. It is known that if the cone $K$ is normal in $X$, then every order-bounded set in $X$ is norm-bounded. The details of cones and their properties appear in Guo and Lakshmikantham [6].

We equip the space $AC(J, \mathbb{R})$ with the order relation $\leq$ with the help of the cone defined by

$$K = \{x \in AC(J, \mathbb{R}) : x(t) \geq 0, \forall t \in J\}.$$  

It is well known that the cone $K$ is positive and normal in $AC(J, \mathbb{R})$. As a result of positivity of the cone $K$ in $AC(J, \mathbb{R})$ we have:

**Lemma 1.** (Dhage [1]). Let $u_1, u_2, v_1, v_2 \in K$ be such that $u_1 \leq v_1$ and $u_2 \leq v_2$. Then $u_1 u_2 \leq v_1 v_2$. For any $a, b \in X = AC(J, \mathbb{R}), a \leq b$, the order interval $[a, b]$ is a set in $X$ given by

$$[a, b] = \{x \in X : a \leq x \leq b\}.$$  

We use the following fixed point theorem of Dhage [1] for proving the existence of extremal solutions of the FIDE (1) under certain monotonicity conditions.

**Theorem 3.** (Dhage [1], Corollary 3.1). Let $K$ be a cone in a Banach algebra $X$ and let $a, b \in X$. Suppose that $A, B : [a, b] \rightarrow K$ are two operators such that

(a) $A$ is Lipschitzian with a Lipschitz constant $\alpha$, 

(b) $B$ is increasing. 

Then $\max (A(x), B(x)) \in K$ satisfies

$$\max (A(x), B(x)) = \max (A(x), B(x)),$$ 

for all $x \in [a, b]$. □
(b) \( B \) is completely continuous,
(c) \( AxBx \in [a, b] \) for each \( x \in [a, b] \), and
(d) \( A \) and \( B \) are nondecreasing.

Further if the cone \( K \) is positive and normal, then the operator equation \( AxBx = x \) has a maximal and a minimal positive solution in \([a, b]\), whenever \( \alpha M < 1 \), where \( M = \| B([a, b]) \| \). We need the following definitions in the sequel.

**Definition 3.** A function \( u \in AC(J, \mathbb{R}) \) is called a lower solution of the IVP (1) on \( J \) if

\[
\left( \frac{u(t)}{f(t, u(\theta(t)))} \right)' \leq g \left( t, u(\mu(t)), \int_0^{\sigma(t)} k(t, s, u(\eta(s))) \, ds \right), \quad \text{a.e } t \in J 
\]

and

\[ u(0) \leq 0. \]

Again a function \( v \in AC(J, \mathbb{R}) \) is called an upper solution of the BVP (1) on \( J \) if

\[
\left( \frac{v(t)}{f(t, v(\theta(t)))} \right)' \geq g \left( t, v(\mu(t)), \int_0^{\sigma(t)} k(t, s, v(\eta(s))) \, ds \right), \quad \text{a.e } t \in J 
\]

and

\[ v(0) \geq 0. \]

**Definition 4.** A solution \( x_M \) of the FIDE(1) is said to be maximal if for any other solution \( x \) to FIDE (1) one has \( x(t) \leq x_M(t), \forall t \in J \). Again a solution \( x_m \) of the FIDE(1) is said to be minimal if \( x_m(t) \leq x(t), \forall t \in J \), where \( x \) is any solution of the FIDE(1) on \( J \).

We consider the following set of assumptions:

- (B_0) \( f : J \times \mathbb{R}^+ \to \mathbb{R}^+ - \{0\}, g : J \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \) and \( x_0 \geq 0 \).
- (B_1) \( g(t, x, y) \) is Carathéodory.
- (B_2) The functions \( f(t, x) \) and \( g(t, x, y) \) are nondecreasing in \( x \) and \( y \) almost everywhere for \( t \in J \).
- (B_3) The FIDE(1) has a lower solution \( u \) and an upper solution \( v \) on \( J \) with \( u \leq v \).

**Remark 2.** Assume that (B_1)–(B_3) hold. Define a function \( h : J \to \mathbb{R}^+ \) by

\[
h(t) = \left| g \left( t, u(\mu(t)), \int_0^{\sigma(t)} k(t, s, u(\eta(s))) \, ds \right) \right| + \left| g \left( t, v(\mu(t)), \int_0^{\sigma(t)} k(t, s, v(\eta(s))) \, ds \right) \right|, \quad \forall t \in J.
\]
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Then $h$ is Lebesgue integrable and
\[
\left| g\left(t, x(t), \int_0^{\sigma(t)} k(t, s, x(\eta(s))) \, ds \right) \right| \leq h(t), \quad \text{a.e. } t \in J, \ \forall x \in [u, v].
\]

**Theorem 4.** Suppose that the assumptions $(A_0)-(A_2)$ and $(B_0)-(B_3)$ hold. Further if
\[
(10) \quad \|\gamma\| \left( \left| \frac{x_0}{f(0, x_0)} \right| + \|h\|_{L^1} \right) < 1,
\]
where $h$ is given in Remark 2, then FIDE (1) has a minimal and a maximal positive solution on $J$.

**Proof.** Now FIDE(1) is equivalent to FIE (4) on $J$. Let $X = AC(J, \mathbb{R})$. Define two operators $A$ and $B$ on $X$ by (6) and (7) respectively. Then FIE (4) is transformed into an operator equation $Ax(t)Bx(t) = x(t)$ in a Banach algebra $X$. Notice that $(B_1)$ implies $A, B : [u, v] \to K$. Since the cone $K$ in $X$ is normal, $[u, v]$ is a norm bounded set in $X$. Now it is shown, as in the proof of Theorem 2, that $A$ is a Lipschitzian with a Lipschitz constant $\|\alpha\|_{L^1}$ and $B$ is completely continuous operator on $[u, v]$. Again the hypothesis $(B_2)$ implies that $A$ and $B$ are nondecreasing on $[u, v]$. To see this, let $x, y \in [u, v]$ be such that $x \leq y$. Then by $(B_2)$,
\[
Ax(t) = f(t, x(\theta(t))) \leq f(t, y(\theta(t))) = Ay(t), \ \forall t \in J,
\]
and
\[
Bx(t) = \frac{x_0}{f(0, x_0)} + \int_0^t g\left(s, x(\mu(s)), \int_0^{\sigma(s)} k(s, \tau, x(\eta(\tau))) \, d\tau \right) \, ds
\]
\[
\leq \frac{x_0}{f(0, x_0)} + \int_0^t g\left(s, y(\mu(s)), \int_0^{\sigma(s)} k(s, \tau, y(\eta(\tau))) \, d\tau \right) \, ds
\]
\[
= By(t), \ \forall t \in J.
\]
So $A$ and $B$ are nondecreasing operators on $[u, v]$. Again Lemma 3.1 and hypothesis $(B_3)$ implies that $u(t) \leq$
\[
\leq [f(t, u(\theta(t))))] \left( \frac{x_0}{f(0, x_0)} + \int_0^t g\left(s, u(\mu(s)), \int_0^{\sigma(s)} k(s, \tau, u(\eta(\tau))) \, d\tau \right) \, ds \right)
\]
\[
\leq [f(t, x(\theta(t))))] \left( \frac{x_0}{f(0, x_0)} + \int_0^t g\left(s, x(\mu(s)), \int_0^{\sigma(s)} k(s, \tau, x(\eta(\tau))) \, d\tau \right) \, ds \right)
\]
\[
\leq [f(t, v(\theta(t))))] \left( \frac{x_0}{f(0, x_0)} + \int_0^t g\left(s, v(\mu(s)), \int_0^{\sigma(s)} k(s, \tau, v(\eta(\tau))) \, d\tau \right) \, ds \right)
\]
\[
\leq v(t)
\]
for all \( t \in J \) and \( x \in [u, v] \). As a result, \( u(t) \leq Ax(t)Bx(t) \leq v(t), \forall t \in J \) and \( x \in [u, v] \). Hence \( Ax B x \in [u, v], \forall x \in [u, v] \).

Again

\[
M = \|BM([u, v])\|
\]
\[
= \sup\{\|Bx\| : x \in [u, v]\}
\leq \sup_{x \in [u, v]} \left\{ \left| \frac{x_0}{f(0, x_0)} \right| + \sup_{t \in J} \int_0^t \left| g(s, x(\mu(s)), \int_0^{\sigma(s)} k(s, \tau, x(\eta(\tau))) d\tau) \right| ds \right\}
\leq \left| \frac{x_0}{f(0, x_0)} \right| + \int_0^a h(s) ds
\leq \left| \frac{x_0}{f(0, x_0)} \right| + \|h\|_{L^1}.
\]

Since \( \|\gamma\| M \leq \|\gamma\| \left( \left| \frac{x_0}{f(0, x_0)} \right| + \|h\|_{L^1} \right) < 1 \), we apply Theorem 3 to FIE (4) to yield that the IVP (1) has a minimal and a maximal positive solution on \( J \). This completes the proof. \( \square \)

REFERENCES


