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GLOBAL PROPERTIES OF SOLUTIONS OF THE FUNCTIONAL DIFFERENTIAL EQUATION $X(T)X'(T) = KX(X(T)),\ 0 < |K| < 1$ *

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Abstract. Maximal solutions of the functional differential equation $x(t)x'(t) = kx(x(t))$ with $0 < |k| < 1$ are considered. It is shown that properties of maximal solutions depend on the sign of the parameter k. The structure of all maximal solutions is completely described for $k \in (-1, 0)$. If $k \in (0, 1)$ then the structure of maximal solutions is described only for such maximal solutions x for which the equation $x(t) - t = 0$ is solvable.

Key Words. Iterative functional differential equation, maximal solution, existence, continuation, global properties, structure of solutions.

AMS(MOS) subject classification. 34K15

1. Introduction. In this paper we consider global properties of solutions for the functional differential equation

$$x(t)x'(t) = kx(x(t)), \tag{1}$$

where $k \neq 0$ is a constant. Equation (1) belongs to the class of iterative functional differential equations in which iterations of the unknown function x appear. Local and global properties of solutions for first order iterative functional differential equations were considered in many papers. Local properties for instance in [4], [5], [7]–[12] and global properties for example in [6],

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[13]–[19]. Equation (1) unlike equations studied in [4]–[19] is a singular functional differential equation since on its left hand side we have $x(t)x'(t)$ and some solution $x(t)$ of this equation may be vanished at a point. We note that equation (1) with $k = 1$ appeared in [1], [2] and [3, p. 239]. G. Barba [1] by equation (1) with $k = 1$ studied curves having certain properties.

Global properties of solutions for equation (1) depend on values of the parameter k. In this paper we consider especially global properties of solutions for equation (1) with $k \in (-1, 0) \cup (0, 1)$. The consideration of global properties of solutions for equation (1) with $|k| \geq 1$ is an open problem.

The paper is organized as follows. In Section 2 we give definitions of a solution and a maximal solution of (1), present some properties of solutions of (1) with $k \neq 0$ (Lemmas 1–3) and $k \in [-1, 0) \cup (0, 1]$ (Lemma 4, Theorem 1) and prove local existence and uniqueness results for problem (1), $x(a) = a$ with $k \in (-1, 0) \cup (0, 1)$ (Lemma 5, Corollary 2). In Section 3 we consider properties of maximal solutions of (1) with $k \in (0, 1)$ and describe the structure of all maximal solutions x for which the equation $x(t) - t = 0$ is solvable (Theorems 2 and 3). Finally, in Section 4, the structure of all maximal solutions of (1) with $k \in (-1, 0)$ is completely described in Theorems 4 and 5.

2. Definitions, auxiliary results.

DEFINITION 1. *We say that x is a solution of* (1) *on an interval J if x has the derivative on J and* (1) *is satisfied for $t \in J$.*

REMARK 1. If x is a solution of (1) on an interval J and $I \subset J$ is an interval such that $x(t) \neq 0$ for $t \in I$, then from the equality $x'(t) = k\frac{x(x(t))}{x(t)}$, $t \in I$, it follows that $x \in C^1(I)$.

DEFINITION 2. *Let x and y be solutions of* (1) *on intervals J and I, respectively. If $J \subset I$, $J \neq I$ and $x(t) = y(t)$ for $t \in J$, then we say that y is a continuation of x. In addition, we say that y is a left* (*resp. right*) *continuation of x if* $\inf\{t : t \in I\} < \inf\{t : t \in J\}$ (*resp.* $\sup\{t : t \in I\} > \sup\{t : t \in J\}$). *A solution x of* (1) *is said to be a maximal solution of* (1) *if x has no continuation.*

REMARK 2. The function $x(t) = 0$ for $t \in \mathbb{R}$ is a maximal solution of (1) on $\mathbb{R}$.

REMARK 3. Let x be a maximal solution of (1) on an interval J. If $x(t) \neq 0$ for $t \in J$ then from the equality $x'(t) = k\frac{x(x(t))}{x(t)}$ it may by concluded that $C^\infty(J)$.

The following lemma is obvious.

LEMMA 1. *Let x be a solution of* (1) *on an interval J. Then*

$$x : J \to J.$$

LEMMA 2. *Let x be a maximal solution of* (1) *on an interval J. Then, for each $m \in \mathbb{R}$, $m \neq 0$, the function $z(t) = \frac{1}{m}x(mt)$ is a maximal solution of* (1) *on the interval $I = \{t : mt \in J\}$.*

Proof. Let $m \in \mathbb{R}$, $m \neq 0$, and let $z(t) = \frac{1}{m}x(mt)$ for $t \in I = \{t : mt \in J\}$. Then

$$z(t)z'(t) = \frac{1}{m}x(mt)x'(mt) = \frac{k}{m}x(x(mt)) = \frac{k}{m}x(mz(t)) = kz(z(t))$$

for $t \in I$. Whence z is a solution of (1) on the interval I. If z is not a maximal solution of (1) then there exists a continuation $\tilde{z}$ of z on an interval I_1, $I \subset I_1$, $I \neq I_1$. Let $J_1 = \{t : \frac{t}{m} \in I_1\}$ and $\tilde{x}(t) = m\tilde{z}(\frac{t}{m})$ for $t \in J_1$. Then $\tilde{x}$ is a solution of (1) on the interval J_1 and since $J \subset J_1$, $J \neq J_1$, we see that $\tilde{x}$ is a continuation of x, which is impossible. □

LEMMA 3. *Let x be a solution of* (1) *on an interval J and $x(t_0) \neq 0$ for some $t_0 \in J$. Then $x'(t_0) \neq 0$.*

Proof. Suppose that $x'(t_0) = 0$. Then

$$x(x(t_0)) = 0 \tag{2}$$

and since $x(t_0) \neq 0$ there is the maximal interval $I \subset J$, $t_0 \in I$, such that $x(t) \neq 0$ for $t \in I$. In addition, from (2) and $x'(x(t_0)) \in \mathbb{R}$ we deduce that

$$|x(t)| \leq S|t - x(t_0)| \tag{3}$$

for $t \in \mathcal{U}(x(t_0), \varepsilon) \cap J$, where $\mathcal{U}(x(t_0), \varepsilon)$ is an ε-neighbourhood of $t = x(t_0)$ and $S = |x'(x(t_0))| + 1$. Let $\delta > 0$ be a positive number such that $|x(t) - x(t_0)| < \varepsilon$ for $t \in \mathcal{U}(t_0, \delta) \cap I$. Then

$$|x(x(t))| \leq S|x(t) - x(t_0)| \tag{4}$$

for $t \in \mathcal{U}(t_0, \delta) \cap I$ which follows from (3). Now, by (4),

$$|x(t) - x(t_0)| = \Big|\int_{t_0}^{t} x'(s)\,ds\Big| = |k|\Big|\int_{t_0}^{t} \frac{x(x(s))}{x(s)}\,ds\Big| \leq |k|S\Big|\int_{t_0}^{t} \frac{|x(s) - x(t_0)|}{|x(s)|}\,ds\Big|$$

and applying the Gronwall lemma, we have $x(t) = x(t_0)$ for $t \in \mathcal{U}(t_0, \delta) \cap I$. Let $I_* \subset J$, $t_0 \in I_*$, be the maximal interval with the property that $x(t) =$

$x(t_0)$ for $t \in I_*$. To prove $I_* = J$, we assume for example that $\alpha = \sup\{t : t \in I_*\} < \sup\{t : t \in J\}$. Then $\alpha \in I_*$, and consequently $x(\alpha) = x(t_0) \neq 0$ and $x'(\alpha) = 0$. If we proceed analogously to the first part of our proof (now with α instead of t_0) we show that $x(t) = x(t_0)$ on a right neighbourhood of $t = \alpha$, contrary to the definition of α. Whence $x(t) = x(t_0)$ for $t \in J$ and then $0 = x(t)x'(t) = kx(x(t)) = kx(t_0)$, contrary to $x(t_0) \neq 0$. □

COROLLARY 1. *Let x be a solution of* (1) *on an interval J and $x(t_1) = x(t_2) = 0$ for some $t_1, t_2 \in J$, $t_1 < t_2$. Then $x(t) = 0$ for $t \in [t_1, t_2]$.*

Proof. If $x(t) \not\equiv 0$ on $[t_1, t_2]$ then there exists $\xi \in (t_1, t_2)$ such that $x(\xi) \neq 0$ and $x'(\xi) = 0$, which contradicts Lemma 3. □

LEMMA 4. *Let $0 < |k| \leq 1$. Let x be a maximal solution of* (1) *on an interval J and $x(t_0) = x'(t_0) = 0$ for some $t_0 \in J$. Then $J = \mathbb{R}$ and $x(t) = 0$ for $t \in \mathbb{R}$.*

Proof. Assume the assertion of our lemma is false. If $x(t) = 0$ for $t \in J$ and $J \neq \mathbb{R}$, then $y = 0$ on $\mathbb{R}$ is a continuation of x, which is impossible. Hence $x(t) \not\equiv 0$ on J. Set $J_1 = \{t : t \in J, x(t) = 0\}$. Then $t_0 \in J_1$, $J_1 \neq J$ and, by Corollary 1, J_1 is an closed interval in J (possibly one point). Let

$$a = \inf\{t : t \in J_1\}, \quad b = \sup\{t : t \in J_1\}.$$

From the equalities $0 = x(t_0)x'(t_0) = kx(x(t_0)) = kx(0)$ we deduce that $0 \in J_1$ and so $-\infty \leq a \leq \min\{0, t_0\}$, $\max\{0, t_0\} \leq b \leq \infty$. Set $\mathcal{M} = \{a, b\} \cap \mathbb{R}$. Clearly $\mathcal{M} \neq \emptyset$ and since x is a maximal solution of (1) on J, we have $\mathcal{M} \subset J$. Even each point of $\mathcal{M}$ is an inner point of J since in the opposite case there exists a continuation of x which is the function $y_a : (-\infty, a) \cup J \to \mathbb{R}$,

$$y_a(t) = \begin{cases} 0 & \text{for } t \in (-\infty, a) \\ x(t) & \text{for } t \in J \end{cases}$$

provided $a \in \mathcal{M}$ and the function $y_b : J \cup (b, \infty) \to \mathbb{R}$,

$$y_b(t) = \begin{cases} x(t) & \text{for } t \in J \\ 0 & \text{for } t \in (b, \infty) \end{cases}$$

provided $b \in \mathcal{M}$.

Assume $b \in \mathcal{M}$. If $a < 0 < b$ then $x(b) = x'(b) = 0$, and so there exists $\varepsilon > 0$, $b + \varepsilon \in J$ such that $|x(t)| < \min\{|a|, b\}$ for $t \in [b, b + \varepsilon]$. Then $x(t)x'(t) = kx(x(t)) = kx(0) = 0$ for $t \in [b, b + \varepsilon]$ which implies $x(t) = 0$ for these t, a contradiction. Hence either $b = 0$ or $b = \infty$ provided $a < 0$. Let $a = 0$. Then $x(t) \neq 0$ for $t \in (-\infty, 0) \cap J$ and $x(0) = x'(0) = 0$. Let

$c > 0$ be such number that $[-c, c] \subset J$ and fix $\mu \in (0,1)$. We now show that $|x(t)| < \mu|t|$ for $t \in [-c, c]\setminus\{0\}$. If not, there exists $\tau \in [-c, c]\setminus\{0\}$ such that $|x(t)| < \mu|t|$ for $t \in (-|\tau|, |\tau|) \setminus \{0\}$ and $|x(\tau)| = \mu|\tau|$ which follows from the equalities $x(0) = x'(0) = 0$. Then $|x(x(t))| < \mu^2|t|$ for $t \in (-|\tau|, |\tau|) \setminus \{0\}$ and so

$$x^2(\tau) \le 2|k|\left|\int_0^\tau |x(x(s))|\,ds\right| < 2|k|\mu^2 \int_0^{|\tau|} s\,ds = |k|\mu^2\tau^2 \le \mu^2\tau^2,$$

contrary to $|x(\tau)| = \mu|\tau|$. Hence $|x(t)| \le \mu|t|$ for $t \in [-c, c]$ and each $\mu \in (0,1)$ which implies $x = 0$ on $[-c, c]$, contrary to $a = 0$. Summarizing, we have proved that $a < 0$ and either $b = 0$ or $b = \infty$. In a similar way we can show that $b > 0$ and either $a = 0$ or $a = -\infty$. Consequently, $a = -\infty$ and $b = \infty$. Then $J_1 = \mathbb{R}$, a contradiction. □.

THEOREM 1. *Let $0 < |k| \le 1$ and $x \ne 0$ be a nontrivial maximal solution of (1) on an interval J. Then $x'(t) \ne 0$ for $t \in J$. In addition,*

a) if $x(t) \ne 0$ for $t \in J$ then $x'(t)\operatorname{sign} k > 0$ for $t \in J$,

b) if $x(t_0) = 0$ for some $t_0 \in J$ then $t_0 = 0$, $x(t) \ne 0$ for $t \in J \setminus \{0\}$ and $k > 0$.

Proof. If $x'(\xi) = 0$ for some $\xi \in J$ then $x(\xi) = 0$ by Lemma 3. Now Lemma 4 gives $J = \mathbb{R}$ and $x = 0$, contrary to our assumption $x \ne 0$. Hence $x'(t) \ne 0$ for $t \in J$.

If $x(t) \ne 0$ for $t \in J$ then from the equality

$$x'(t) = k\frac{x(x(t))}{x(t)} \tag{5}$$

we deduce that $x'(t)\operatorname{sign} k > 0$ for $t \in J$.

Assume $x(t_0) = 0$ for some $t_0 \in J$. Then $x'(t_0) \ne 0$. If there exists $t_1 \ne t_0$ such that $x(t_1) = 0$ then $x(t) = 0$ on the interval with the end points t_0 and t_1 by Corollary 1. Whence $x(t_0) = x'(t_0) = 0$, contrary to Lemma 4. If $t_0 \ne 0$ then $0 = x(t_0)x'(t_0) = kx(x(t_0)) = kx(0)$, and so $x(0) = 0$, a contradiction. Hence $t_0 = 0$ and $x(t) \ne 0$ for $t \in J\setminus\{0\}$. Now from $x'(0) \ne 0$ we see that $x(t)\operatorname{sign} x'(0) < 0$ for $t \in (-\infty, 0) \cap J$ and $x(t)\operatorname{sign} x'(0) > 0$ for $t \in (0, \infty) \cap J$ which imply $x(x(t))\operatorname{sign} t > 0$ for $t \in J \setminus \{0\}$. If $k < 0$ then $x(t)x'(t)\operatorname{sign} t < 0$ for $t \in J \setminus \{0\}$, and so $(x^2(t))' > 0$ for $t \in (-\infty, 0) \cap J$, $(x^2(t))' < 0$ for $t \in (0, \infty) \cap J$, contrary to $x(0) = 0$. Hence $k > 0$. □

LEMMA 5. *Let $0 < |k| < 1$, $a > 0$ and*

$$0 < \varepsilon < a \min\left\{1,\ \frac{1}{2\sqrt{|k|}},\ \frac{1}{\sqrt{|k|}} - 1,\ \frac{1 - |k|}{4|k|}\right\}. \tag{6}$$

Then there exists the unique solution x of (1) *on the interval $J = [a-\varepsilon, a+\varepsilon]$ such that $x(a) = a$.*

Proof. Let $\mathbf{X}$ be the Banach space of continuous functions $x : J \to \mathbb{R}$ with the norm $\|x\| = \max\{|x(t)| : t \in J\}$ where $J = [a-\varepsilon, a+\varepsilon]$. Let

$$\Omega = \{x : x \in \mathbf{X},\ x(a) = a,\ |x(t_1) - x(t_2)| \le |t_1 - t_2| \text{ for } t_1, t_2 \in J\}.$$

Then Ω is a closed bounded subset of $\mathbf{X}$ and, for each $x, y \in \Omega$ and $t \in J$, we have $x : J \to J$, $a - \varepsilon \le x(x(t)) \le a + \varepsilon$,

$$\begin{aligned} |x(x(t)) - y(y(t))| &\le |x(x(t)) - x(y(t))| + |x(y(t)) - y(y(t))| \\ &\le |x(t) - y(t)| + \|x - y\| \le 2\|x - y\| \end{aligned} \tag{7}$$

and

$$\begin{aligned} a^2 + 2k\int_a^t x(x(s))\,ds &\ge a^2 - 2|k|(a+\varepsilon)\varepsilon > a^2 - 4|k|a\varepsilon \\ &> a^2 - 4|k|a^2\frac{1-|k|}{4|k|} = |k|a^2. \end{aligned} \tag{8}$$

Define the operator $\mathcal{L} : \Omega \to \mathbf{X}$ by the formula

$$(\mathcal{L}x)(t) = \sqrt{a^2 + 2k\int_a^t x(x(s))\,ds}.$$

By (8),

$$\begin{aligned} &(\mathcal{L}x)(t_1) + (\mathcal{L}x)(t_2) \\ &\quad= \sqrt{a^2 + 2k\int_a^{t_1} x(x(s))\,ds} + \sqrt{a^2 + 2k\int_a^{t_2} x(x(s))\,ds} \\ &\quad\ge 2\sqrt{|k|}a, \end{aligned} \tag{9}$$

$$\begin{aligned} &(\mathcal{L}x)(t) + (\mathcal{L}y)(t) \\ &\quad= \sqrt{a^2 + 2k\int_a^{t} x(x(s))\,ds} + \sqrt{a^2 + 2k\int_a^{t} y(y(s))\,ds} \\ &\quad\ge 2\sqrt{|k|}a \end{aligned} \tag{10}$$

for $x, y \in \Omega$ and $t, t_1, t_2 \in J$. Hence from the inequalities (cf. (7), (9) and (10))

$$|(\mathcal{L}x)(t_1) - (\mathcal{L}x)(t_2)| = \frac{|((\mathcal{L}x)(t_1))^2 - ((\mathcal{L}x)(t_2))^2|}{(\mathcal{L}x)(t_1) + (\mathcal{L}x)(t_2)}$$

$$\begin{aligned}
&\leq \frac{2|k|}{2\sqrt{|k|a}}\Big|\int_{t_1}^{t_2} |x(x(t))|\,ds\Big| \\
&\leq \frac{\sqrt{|k|}}{a}(a+\varepsilon)|t_1-t_2| \leq |t_1-t_1|,
\end{aligned}$$

$$\begin{aligned}
|(\mathcal{L}x)(t)-(\mathcal{L}y)(t)| &= \frac{|((\mathcal{L}x)(t))^2-((\mathcal{L}y)(t))^2|}{(\mathcal{L}x)(t)+(\mathcal{L}y)(t)} \\
&\leq \frac{2|k|}{2\sqrt{|k|a}}\Big|\int_a^t |x(x(t))-y(y(t))|\,ds\Big| \\
&\leq \frac{2\sqrt{|k|}\varepsilon}{a}\|x-y\|
\end{aligned}$$

where $x,y\in\Omega$ and $t,t_1,t_2\in J$, we deduce that $\mathcal{L}:\Omega\to\Omega$ and $\mathcal{L}$ is contractive since $\frac{2\sqrt{|k|}\varepsilon}{a}<1$. By the Banach fixed point theorem, there exists a unique fixed point x of the operator $\mathcal{L}$. Then

$$x(t)=\sqrt{a^2+2k\int_a^t x(x(s))\,ds}, \quad t\in J,$$

and we see that x is the unique solution of (1) on the interval J satisfying $x(a)=a$. □

COROLLARY 2. *Let $0<|k|<1$, $a<0$ and let ε satisfy inequalities* (6). *Then there exists the unique solution x of* (1) *on the interval $J=[a-\varepsilon,a+\varepsilon]$ such that $x(a)=a$.*

Proof. By Lemma 5, there exists the unique solution y of (1) on the interval $I=[-a-\varepsilon,-a+\varepsilon]$ such that $y(-a)=-a$. Set $x(t)=-y(-t)$ for $t\in J=[a-\varepsilon,a+\varepsilon]$. Then x is a solution of (1) on the interval J which follows from the proof of Lemma 2 with $m=-1$, and $x(a)=a$. To prove the uniqueness of x, assume that x_1 is another solution of (1) on J and $x_1(a)=a$. Then $y_1(t)=-x_1(-t)$, $t\in I$, is a solution of (1) on I and $y_1(-a)=-a$. Hence $y=y_1$ by Lemma 5, and so $x=x_1$. □

3. Equation $x(t)x'(t)=kx(x(t))$, $0<k<1$.

LEMMA 6. *Let $0<k<1$ and $x>0$ be a maximal solution of* (1) *on an interval J. Then $J=(A,\infty)$ with an $A\in(-\infty,0]$ and $\lim_{t\to A^+}x(t)=0$, $\lim_{t\to\infty}x(t)=\infty$. Moreover, for $A<0$, we have $\lim_{t\to A^+}x'(t)=\infty$.*

Proof. Our assumptions $x > 0$ and $0 < k < 1$ imply $x' > 0$ on J. Set

$$A = \inf\{t : t \in J\}, \quad B = \sup\{t : t \in J\}.$$

Then $A \geq -\infty$ and Lemma 1 gives $0 < B \leq \infty$. Suppose $B < \infty$. We are going to show that $B \in J$. Let $B \notin J$. Since x is increasing on J, there exists $\lim_{t \to B^-} x(t) = \beta$ and $0 < \beta \leq B$. Define $y : J \cup \{B\} \to \mathbb{R}$ by

$$y(t) = \begin{cases} x(t) & \text{for } t \in J \\ \beta & \text{for } t = B. \end{cases}$$

Taking the limit as $\varepsilon \to B^-$ in the equalities

$$y(\varepsilon) - y(t) = x(\varepsilon) - x(t) = k \int_t^\varepsilon \frac{x(x(s))}{x(s)}\, ds = k \int_t^\varepsilon \frac{y(y(s))}{y(s)}\, ds, \qquad \varepsilon, t \in J,$$

we get

$$\beta - y(t) = k \int_t^B \frac{y(y(s))}{y(s)}\, ds, \qquad t \in J.$$

Hence

$$y'(B) = \lim_{t \to B^-} \frac{\beta - y(t)}{B - t} = \lim_{t \to B^-} \frac{k}{B - t} \int_t^B \frac{y(y(s))}{y(s)}\, ds = k \frac{y(y(B))}{y(B)},$$

and consequently y is a solution of (1) on the interval $J \cup \{B\}$. Then y is a right continuation of x, a contradiction. Whence $B \in J$. If $x(B) < B$ then the problem

$$y' = k \frac{x(y)}{y}, \tag{11}$$

$$y(B) = x(B) \tag{12}$$

has a unique solution y on a neighbourhood $\mathcal{U}$ of the point $t = B$ since $x \in C^1(J)$. On the other hand x is a solution of problem (11), (12) on J, and so $x(t) = y(t)$ for $t \in J \cap \mathcal{U}$. One can easy verify that the function $z : J \cup \mathcal{U} \to \mathbb{R}$,

$$z(t) = \begin{cases} x(t) & \text{for } t \in J \\ y(t) & \text{for } t \in \mathcal{U} \setminus J \end{cases}$$

is a right continuation of x, which is impossible. Hence $x(B) = B$ but in this case there exists a right continuation of x by Lemma 5, which is again impossible. We have proved that $B = \infty$.

With respect to A we consider four cases.

Case 1. Let $A = -\infty$ and set $\alpha = \lim_{t\to-\infty} x(t)$. If $\alpha = 0$, we have $\lim_{t\to-\infty} x(t)x'(t) = k\lim_{t\to-\infty} x(x(t)) = kx(0) > 0$. Hence $\lim_{t\to-\infty} x'(t) = \infty$, which is impossible. If $\alpha > 0$ then $x(\alpha) > 0$ and

$$\alpha \lim_{t\to-\infty} x'(t) = k \lim_{t\to-\infty} x(x(t)) = kx(\alpha) > 0.$$

Whence $\lim_{t\to-\infty} x'(t) = \frac{kx(\alpha)}{\alpha} > 0$, which is impossible.

Case 2. Let $-\infty < A < 0$. Assume $J = [A, \infty)$. Since $x > 0$ by the assumption of our lemma, $x(A) > 0$ and so the problem (11), (12) with A instead of B has a unique solution y in a neighbourhood $\mathcal{V}$ of the point $t = A$. We know that x is a solution of the above problem on J, hence $x(t) = y(t)$ for $t \in J \cap \mathcal{V}$. Consequently, the function $z : J \cup \mathcal{V} \to \mathbb{R}$,

$$z(t) = \begin{cases} x(t) & \text{for } t \in J \\ y(t) & \text{for } t \in \mathcal{V} \setminus J \end{cases}$$

is a left continuation of x, which is impossible. Therefore $J = (A, \infty)$. Let $\alpha = \lim_{t\to A^-} x(t)$. If $\alpha > 0$ then the function $z : [A, \infty) \to \mathbb{R}$,

$$z(t) = \begin{cases} x(t) & \text{for } t \in (A, \infty) \\ \alpha & \text{for } t = A \end{cases}$$

is a left continuation of x (see the first part of the proof of our lemma), a contradiction. Whence $\alpha = 0$ and then from the equalities $\lim_{t\to A^+} x(t)x'(t) = k\lim_{t\to A^+} x(x(t)) = kx(0) > 0$ it follows that $\lim_{t\to A^+} x'(t) = \infty$.

Case 3. Let $A = 0$. Suppose $J = [0, \infty)$. Then $x(0) > 0$ and applying the procedure as in the first part of *Case 2* we can show that there exists a left continuation of x, which is impossible. Therefore $J = (0, \infty)$. Let $\alpha = \lim_{t\to 0^+} x(t)$. Then $\alpha \geq 0$ and if $\alpha > 0$ then the function $u : [0, \infty) \to \mathbb{R}$,

$$u(t) = \begin{cases} x(t) & \text{for } t \in (0, \infty) \\ \alpha & \text{for } t = 0 \end{cases}$$

is a left continuation of x, which is impossible. Whence $\lim_{t\to 0^+} x(t) = 0$.

Case 4. Let $A > 0$. Suppose $J = [A, \infty)$. Then $x(t) \geq A$ for $t \in [A, \infty)$ by Lemma 1. If $x(A) = A$, then there exists a left continuation of x by

Lemma 5, a contradiction. If $x(A) > A$, the problem (11), (12) with A instead of B has a unique solution on a neighbourhood of the point $t = A$ and as in the first part of *Case 2* we can prove that there exists a left continuation of x, which is impossible. Hence $J = (A, \infty)$. Let $\alpha = \lim_{t\to A^+} x(t)$. Then the function $w : [A, \infty) \to \mathbb{R}$,

$$w(t) = \begin{cases} x(t) & \text{for } t \in (A, \infty) \\ \alpha & \text{for } t = A \end{cases}$$

is a left continuation of x, which is again impossible.

We have proved that $J = (A, \infty)$ where $A \in (-\infty, 0]$. It remains to verify that $\lim_{t\to\infty} x(t) = \infty$. We know that x is increasing on J. Suppose that $\lim_{t\to\infty} x(t) = \gamma < \infty$. Then

$$\lim_{t\to\infty} x'(t) = k \lim_{t\to\infty} \frac{x(x(t))}{x(t)} = k\frac{x(\gamma)}{\gamma} > 0,$$

contrary to $\lim_{t\to\infty} x(t) = \gamma$. Hence $\lim_{t\to\infty} x(t) = \infty$. □

LEMMA 7. *Let $0 < k < 1$ and $x \neq 0$ be a nontrivial maximal solution of* (1) *on an interval J. Then $x(t) \neq 0$ for $t \in J$ and there exists $A < 0$ such that $J = (A, \infty)$ provided $x > 0$ on J and $J = (-\infty, -A)$ provided $x < 0$ on J.*

Proof. We first show that $x(t) \neq 0$ for $t \in J$. If not, there exists $t_0 \in J$ such that $x(t_0) = 0$. Then $t_0 = 0$ and $x'(0) \neq 0$ by Theorem 1 and Lemma 4. Since $x'(0) = \lim_{t\in J, t\to 0} \frac{x(t)}{t} = \lim_{t\in J, t\to 0} \frac{x(x(t))}{x(t)} = \frac{1}{k} \lim_{t\in J, t\to 0} x'(t)$, there exists $\varepsilon > 0$ such that $0 < \frac{x'(t)}{kx'(0)} < \frac{2}{1+k}$ for $t \in [-\varepsilon, \varepsilon] \cap J$. Then for these t we have

$$|x(t)| = \Big| \int_0^t x'(s)\, ds \Big| \leq \frac{2k|x'(0)|}{1+k}|t|,$$

and consequently

$$|x'(0)| = \lim_{t\in J, t\to 0} \Big|\frac{x(t)}{t}\Big| \leq \frac{2k|x'(0)|}{1+k}.$$

Hence $1 < \frac{2k}{1+k}$, a contradiction. Therefore $x(t) \neq 0$ for $t \in J$. If $x > 0$ on J then, by Lemma 6, there exists $A_+ \leq 0$ such that $J = (A_+, \infty)$. Let $x < 0$ on J. Then the positive function $y(t) = -x(-t)$, $t \in I = \{t : -t \in J\}$ is a maximal solution of (1) on I by Lemma 2 (with $m = -1$). So $I = (A_-, \infty)$ with some $A_- \leq 0$ and we have $J = (-\infty, -A_-)$.

We have proved the assertions of our lemma with some $A \le 0$. Assume that $A = 0$. Without loss of generality we can assume that $x > 0$ on $(0,\infty)$. Then $\lim_{t\to 0^+} x(t) = 0$ by Lemma 6. The next part of the proof is divided into three cases.

Case 1. Let $x(t) < t$ for $t \in (0,\infty)$. Define the continuous function $\tilde{x} : [0,\infty) \to [0,\infty)$ by the formula

$$\tilde{x}(t) = \begin{cases} 0 & \text{for } t = 0 \\ x(t) & \text{for } t \in (0,\infty) \end{cases}$$

and set

$$\mu_0 = \min\{\mu : \mu \in [0,1],\ \tilde{x}(t) \le \mu t \text{ for } t \in [0,1]\}.$$

Then $\tilde{x}(t) \le \mu_0 t$ for $t \in [0,1]$ and $x > 0$ on $(0,\infty)$ implies $\mu_0 > 0$. Since $(x^2(t))' = 2kx(x(t)) \le 2k\mu_0^2 t$ for $t \in (0,1]$, integrating the inequality $(x^2(t))' \le 2k\mu_0^2 t$ from $\varepsilon \in (0,t)$ to t and letting $\varepsilon \to 0^+$ in the obtained inequality, we have $x^2(t) \le k\mu_0^2 t^2$ for $t \in (0,1]$. So $\tilde{x}(t) \le \sqrt{k}\mu_0 t$ for $t \in [0,1]$, contrary to the definition of μ_0.

Case 2. Let $x(\tau) = \tau$ for some $\tau \in (0,\infty)$. Then $x'(\tau) = k < 1$ and so $x(t) < t$ on a right neighbourhood of the point $t = \tau$. If there exists $\xi > \tau$ such that $x(\xi) = \xi$ and $x(t) < t$ for $t \in (\tau,\xi)$, then $x'(\xi) \ge 1$, contrary to $x'(\xi) = k$. Therefore the equation $x(t) = t$ has the unique solution $t = \tau$ and $x(t) > t$ for $t \in (0,\tau)$ and $x(t) < t$ for $t \in (\tau,\infty)$. Define $F : (0,\infty) \to \mathbb{R}$ by

$$F(t) = \int_t^{x(t)} \frac{1}{x(s)}\, ds. \tag{13}$$

Then

$$F'(t) = \frac{x'(t)}{x(x(t))} - \frac{1}{x(t)} = \frac{k-1}{x(t)} < 0, \quad t \in (0,\infty) \tag{14}$$

and, by the mean value theorem,

$$F(t) = \frac{1}{x(\xi)}(x(t) - t) < \frac{1}{x(t)}(x(t) - t) = 1 - \frac{t}{x(t)} < 1, \quad t \in (0,\tau)$$

since $\xi \in (t, x(t))$, x is increasing on $(0,\infty)$ and $x(t) - t > 0$ for $t \in (0,\tau)$. Hence F is decreasing on $(0,\infty)$ and bounded on $(0,\tau)$. Integrating the equality $F'(t) = \frac{k-1}{x(t)}$ from t_0 to t, we get

$$\int_t^{x(t)} \frac{1}{x(s)}\, ds - \int_{t_0}^{x(t_0)} \frac{1}{x(s)}\, ds = (k-1)\int_{t_0}^{t} \frac{1}{x(s)}\, ds \tag{15}$$

for $t_0, t \in (0, \infty)$. From (15) and the boundedness of F on $(0, \tau)$ it follows that $\int_0^\tau \frac{1}{x(s)}\, ds$ is convergent, and so $\lim\limits_{t_0 \to 0^+} \int_{t_0}^{x(t_0)} \frac{1}{x(s)}\, ds = 0$. Letting $t_0 \to 0^+$ in (15) yields

$$\int_t^{x(t)} \frac{1}{x(s)}\, ds = (k-1) \int_0^t \frac{1}{x(s)}\, ds \quad \text{for } t \in (0, \infty). \tag{16}$$

Setting $t = \tau$ in (16) gives

$$0 = \int_\tau^\tau \frac{1}{x(s)}\, ds = (k-1) \int_0^\tau \frac{1}{x(s)}\, ds,$$

contrary to $\int_0^\tau \frac{1}{x(s)}\, ds > 0$.

Case 3. Let $x(t) > t$ for $t \in (0, \infty)$. Let F be defined by (13). Then (16) holds and so

$$\int_0^{x(t)} \frac{1}{x(s)}\, ds = k \int_0^t \frac{1}{x(s)}\, ds, \qquad t \in (0, \infty),$$

contrary to $x(t) > t$ for $t \in (0, \infty)$ and $k \in (0, 1)$. □

Remark 4. Let $0 < k < 1$ and $x > 0$ be a maximal solution of (1) on an interval (A, ∞), $A < 0$. Arguing as in *Case 2* of the proof of Lemma 7 we can verify that the equation $x(t) = t$ has at most one solution. In addition, if $x(a) = a$ for some $a \in (A, \infty)$, then $x(t) > t$ for $t \in (A, a)$ and $x(t) < t$ for $t \in (a, \infty)$.

Let $0 < k < 1$. For each positive maximal solution x of (1) on an interval (A, ∞) define the function $\Phi(t; x) : (A, \infty) \to \mathbb{R}$ by

$$\Phi(t; x) = \int_t^{x(t)} \frac{1}{x(t)}\, ds. \tag{17}$$

Lemma 8. *Let $0 < k < 1$ and x be a positive solution of* (1) *on an interval (A, ∞). Then $\Phi(t; x)$ is decreasing on (A, ∞) and $\lim_{t \to A^+} \Phi(t; x) < \infty$. If $x(t) > t$ for $t \in (A, \infty)$ then $\lim_{t \to \infty} \Phi(t; x) = 0$ and if $\Phi(a; x) = 0$ for some $a \in (A, \infty)$ then $\lim_{t \to \infty} \Phi(t; x) = -\infty$.*

Proof. Since $A < 0$ by Lemma 7, $x(t) > t$ on a right neighbourhood I of the point $t = A$. From the equality (see (14))

$$\Phi'(t; x) = \frac{x'(t)}{x(x(t))} - \frac{1}{x(t)} = \frac{k-1}{x(t)} < 0, \quad t \in (A, \infty),$$

we deduce that $\Phi(\cdot;x)$ is decreasing on (A,∞) and $\Phi(\cdot;x) > 0$ on I. By Lemma 6, $\lim_{t\to A^+} x(t) = 0$ and consequently $\lim_{t\to A^+} x(x(t)) = x(0) > 0$. Hence $x(x(t)) > x(0)$ on (A,∞) and from

$$x^2(t) = 2k\int_A^t x(x(s))\,ds > 2kx(0)(t-A), \quad t\in(A,\infty),$$

we deduce that

$$\frac{1}{x(t)} < \frac{1}{\sqrt{2kx(0)(t-A)}}, \quad t\in(A,\infty).$$

Then $\int_A^0 \frac{1}{x(t)}\,ds$ is convergent and so

$$\lim_{t\to A^+}\Phi(t;x) = \int_A^0 \frac{1}{x(s)}\,ds < \infty.$$

Let $x(t) > t$ for $t\in(A,\infty)$ and set $\beta = \lim_{t\to\infty}\Phi(t;x)$ (≥ 0). If $\beta > 0$ then necessarily $\int_A^\infty \frac{1}{x(s)}\,ds = \infty$ and we have

$$\lim_{t\to\infty}\frac{\Phi(t;x)}{\displaystyle\int_A^t \frac{1}{x(s)}\,ds} = 0.$$

On the other hand, by the L'Hospital rule

$$\lim_{t\to\infty}\frac{\Phi(t;x)}{\displaystyle\int_A^t \frac{1}{x(s)}\,ds} = k-1 < 0, \tag{18}$$

a contradiction. Hence $\lim_{t\to\infty}\Phi(t;x) = 0$.

Assume now that $\Phi(a;x) = 0$ for some $a\in(A,\infty)$. Then $x(a) = a$ and $x(t) < t$ for $t\in(a,\infty)$ (see Remark 4). Hence $\frac{1}{x(t)} > \frac{1}{t}$ for $t\in(a,\infty)$ which implies that $\int_a^\infty \frac{1}{x(s)}\,ds = \infty$ and then (18) shows that $\lim_{t\to\infty}\Phi(t;x) = -\infty$. □

From Lemma 8 it follows that the set of positive maximal solutions of (1) is the union of two sets. The first set $\mathcal{A}_+$ is formed by solutions x of (1) with $\Phi(t;x)$ vanishing at a point. Solutions x of (1) for which $\Phi(t;x)$ is positive (on the domain of x) belong to the second set. Next lemma is very important for a description of a structure of the set $\mathcal{A}_+$.

LEMMA 9. *Let $0<k<1$. Then there exists the unique maximal solution x of (1) satisfying $x(1) = 1$. This solution x is defined on an interval (A,∞), $A<0$, and has the following properties:*

a) $x(t) > 0$, $x'(t) > 0$ *for* $t \in (A, \infty)$ *and*

$$\lim_{t\to A^+} x(t) = 0, \ \lim_{t\to A^+} x'(t) = \infty, \ \lim_{t\to\infty} x(t) = \infty,$$

b) $x(t) > t$ *and* $x'(t) > k$ *for* $t \in (A, 1)$,
c) $x(t) < t$ *and* $x'(t) < k$ *for* $t \in (1, \infty)$,
d) $x''(t) < 0$ *for* $t \in (A, \infty)$ *and* $\lim_{t\to\infty} x'(t) = 0$.

Proof. By Lemma 5, there exists the unique solution y of (1) on an interval $[1-\varepsilon, 1+\varepsilon]$, $\varepsilon > 0$, and Lemma 7 shows that there exists a maximal solution x of (1) on an interval (A, ∞), $A < 0$, which is a continuation of y. The property a) of our lemma follows from Theorem 1 and Lemmas 6 and 7. Remark 4 gives that $x(t) > t$ for $t \in (A, 1)$ and $x(t) < t$ for $t \in (1, \infty)$. Therefore $x(x(t)) > x(t)$ for $t \in (A, 1)$ and $x(x(t)) < x(t)$ for $t \in (1, \infty)$ and then from $x'(t) = k\frac{x(x(t))}{x(t)}$ we deduce that $x'(t) > k$ for $t \in (A, 1)$ and $x'(t) < k$ for $t \in (1, \infty)$.

From the equalities (see Remark 3)

$$(x'(t))^2 + x(t)x''(t) = kx'(t)x'(x(t)) \quad \text{for } t \in (A, \infty) \tag{19}$$

and $x(1) = 1$, $x'(1) = k$ we see that $x''(1) = k^2(k-1) < 0$. We first assume that $x''(\xi) = 0$ for a $\xi \in (A, 1)$ and $x'' < 0$ on $(\xi, 1]$. Then x' is decreasing on $[\xi, 1]$ and since $x(t) > t$ for $t \in (A, 1)$, we have $x'(x(\xi)) < x'(\xi)$. Now setting $t = \xi$ in (19) we get $x'(\xi) = kx'(x(\xi)) < kx'(\xi)$. Therefore $1 < k$, a contradiction. Suppose $x''(\eta) = 0$ for an $\eta \in (1, \infty)$ and $x'' < 0$ on $[1, \eta)$. Since $x(t) < t$ for $t \in (1, \infty)$, we have $x''(x(\eta)) < 0$. Differentiating (19), we get

$$3x'(t)x''(t) + x(t)x'''(t) = kx''(t)x'(x(t)) + k(x'(t))^2x''(x(t)), \quad t \in (A, \infty).$$

Setting $t = \eta$ in the last equality, $x(\eta)x'''(\eta) = k(x'(\eta))^2x''(x(\eta)) < 0$. Then $x'''(\eta) < 0$, which is impossible. We have proved that $x'' < 0$ on (A, ∞). Consequently, x' is decreasing on (A, ∞) and let $\lim_{t\to\infty} x'(t) = \gamma \, (\geq 0)$. We conclude from the equality $x(x(t)) = x(t) + x'(\tau(t))(x(t) - t)$ for $t \in (1, \infty)$ where $x(t) < \tau(t) < t$ that $x(t)x'(t) = k[x(t) + x'(\tau(t))(x(t) - t)]$ and

$$x'(t) = k\Big[1 + x'(\tau(t))\Big(1 - \frac{t}{x(t)}\Big)\Big]$$

for $t \in (1, \infty)$. Assume $\gamma > 0$. Then

$$\lim_{t\to\infty} \frac{t}{x(t)} = \lim_{t\to\infty} \frac{1}{x'(t)} = \frac{1}{\gamma}$$

by the L'Hospital rule and $\lim_{t\to\infty} x'(\tau(t)) = \lim_{t\to\infty} x'(t) = \gamma$. Consequently,

$$\gamma = \lim_{t\to\infty} x'(t) = k \lim_{t\to\infty} \Big[1 + x'(\tau(t))\Big(1 - \frac{t}{x(t)}\Big)\Big] = k\Big[1 + \gamma\Big(1 - \frac{1}{\gamma}\Big)\Big] = k\gamma,$$

which is impossible. Hence $\gamma = 0$.

It remains to prove that x is determined by y uniquely. Let w be a maximal solution of (1) on an interval (B, ∞) which is a continuation of y and let $x \neq w$. Of course, $B < 0$. Since $x(t) = w(t)$ for $t \in [1-\varepsilon, 1+\varepsilon]$, there exists $\xi \in (A, 1-\varepsilon) \cup (1+\varepsilon, \infty)$ such that $x(\xi) \neq w(\xi)$. First assume $\xi > 1+\varepsilon$. Then there exists $\xi_1 \in [1+\varepsilon, \xi)$ such that $x(t) = w(t)$ for $t \in [1, \xi_1]$ and $\max\{|x(s) - w(s)| : \xi_1 \leq s \leq t\} > 0$ for $t \in (\xi_1, \xi]$. Without loss of generality we can assume that $w(\xi) \leq \xi_1$ since $w(t) < t$ for $t > 1$. Set

$$R(t) = \max\{|x(s) - w(s)| : \xi_1 \leq s \leq t\} \quad \text{for } t \in [\xi_1, \xi].$$

Then R is continuous, $R(\xi_1) = 0$ and $R(t) > 0$ for $t \in (\xi_1, \xi]$. We see that

$$\begin{aligned} |x(x(t)) - w(w(t))| &\leq |x(x(t)) - x(w(t))| + |x(w(t)) - w(w(t))| \\ &= x'(\eta)|x(t) - w(t)| \leq k|x(t) - w(t)| \end{aligned}$$

for $t \in [\xi_1, \xi]$ and η lying between $x(t)$ and $w(t)$ since $x' < k$ on $(1, \infty)$ and $x(w(t)) - w(w(t)) = 0$ for $t \in [\xi_1, \xi]$ which follows from $w(\xi) \leq \xi_1$. Then (for $t \in [\xi_1, \xi]$)

$$x(t) + w(t) \geq 2x(\xi_1),$$

$$|x^2(t) - w^2(t)| = 2k\Big|\int_{\xi_1}^t (x(x(s)) - w(w(s)))\,ds\Big| \leq 2k^2 \int_{\xi_1}^t |x(s) - w(s)|\,ds,$$

and consequently

$$|x(t) - w(t)| = \frac{|x^2(t) - w^2(t)|}{x(t) + w(t)} \leq \frac{k^2}{x(\xi_1)} \int_{\xi_1}^t |x(s) - w(s)|\,ds \leq \frac{k^2}{x(\xi_1)} R(t)(t - \xi_1).$$

Hence

$$R(t) \leq \frac{k^2}{x(\xi_1)} R(t)(t - \xi_1), \quad t \in [\xi_1, \xi]$$

and then

$$1 \leq \frac{k^2}{x(\xi_1)}(t - \xi_1)$$

for $t \in (\xi_1, \xi]$, which is impossible.

Let $\xi < 1-\varepsilon$. Then there exists $\xi_2 \in (\xi, 1-\varepsilon]$ such that $x(t) = w(t)$ for $t \in [\xi_2, 1]$ and $\max\{|x(s) - w(s)| : t \le s \le \xi_2\} > 0$ for $t \in [\xi, \xi_2)$. Without loss of generality we can assume that $w(\xi) \ge \xi_2$ since $w(t) > t$ for $t \in (B, 1)$. Set

$$c = \max\{x'(t) : t \in [\min\{x(\xi), w(\xi)\}, 1]\}$$

and

$$S(t) = \max\{|x(s) - w(s)| : t \le s \le \xi_2\} \quad \text{for } t \in [\xi, \xi_2].$$

Then $S(\xi_2) = 0$ and $S > 0$ on $[\xi, \xi_2)$. From the relations

$$x(t) + w(t) \ge \min\{x(\xi), w(\xi)\} = L,$$

$$|x(x(t)) - w(w(t))| = |x(x(t)) - x(w(t))| = x'(\tau)|x(t) - w(t)| \le c|x(t) - w(t)|$$

and

$$|x^2(t) - w^2(t)| = 2k \int_t^{\xi_2} |x(x(s)) - w(w(s))|\, ds \le 2ck \int_t^{\xi_2} |x(s) - w(s)|\, ds$$

for $t \in [\xi, \xi_2]$, where τ lies between $x(t)$ and $w(t)$, we deduce that

$$|x(t) - w(t)| \le \frac{2ck}{L} \int_t^{\xi_2} |x(s) - w(s)|\, ds \le \frac{2ck}{L} S(t)(\xi_2 - t)$$

and $S(t) \le \frac{2ck}{L} S(t)(\xi_2 - t)$ for $t \in [\xi, \xi_2]$. Then

$$1 \le \frac{2ck}{L}(\xi_2 - t) \quad \text{for } t \in [\xi, \xi_2),$$

a contradiction. Hence $x = y$. □

Let $0 < k < 1$. Lemma 7 shows that any nontrivial maximal solution of (1) is either positive or negative on its interval of definition. Applying Lemma 2 we can determine the set of all negative positive solutions of (1) by the set of all positive maximal solutions of (1). In addition, by Lemma 8, for each positive maximal solution x of (1) on an interval (A, ∞), the function Φ defined by (17) satisfies either $\lim_{t\to\infty} \Phi(t; x) = -\infty$ or $\lim_{t\to\infty} \Phi(t; x) = 0$ accordingly $x(a) = a$ for some $a \in (A, \infty)$ or $x(t) > t$ for $t \in (A, \infty)$.

We recall that by $\mathcal{A}_+$ we denoted the set of positive maximal solutions of (1) for which $\lim_{t\to\infty} \Phi(t; x) = -\infty$, that is $\mathcal{A}_+$ is the set of positive maximal solutions x of (1) for which the equation $x(t)-t=0$ has a (unique) solution. Analogously, let $\mathcal{A}_-$ denote the set of negative maximal solutions x of (1) for which the equation $x(t)-t=0$ is solvable. Finally, let x_+ be the unique maximal solution of (1) on (A_+,∞), $A_+<0$, satisfying the condition $x_+(1)=1$. The existence and uniqueness of x_+ follows from Lemma 9. In the next Theorems 2 and 3 we will describe the structure of the sets $\mathcal{A}_+$ and $\mathcal{A}_-$.

THEOREM 2. *Let* $0<k<1$ *and set*

$$x_a(t)=ax_+\Big(\frac{t}{a}\Big) \quad \text{for } t\in(aA_+,\infty),\ a>0. \tag{20}$$

Then $\mathcal{A}_+=\{x_a : a\in(0,\infty)\}$.

Proof. Set $\mathcal{B}=\{x_a : a\in(0,\infty)\}$. Fix $a\in(0,\infty)$. By Lemma 2, x_a defined by (20) is a positive maximal solution of (1) on (aA_+,∞) and $x_a(a)=ax_+(1)=a$. Hence $x_a\in\mathcal{A}_+$ and so $\mathcal{B}\subset\mathcal{A}_+$. Let $y\in\mathcal{A}_+$. Then y is a positive maximal solution of (1) on an (B,∞) and $y(b)=b$ for a (unique) $b\in(B,\infty)$. Clearly, $b>0$. Now the function $\tilde{x}(t)=\frac{1}{b}y(bt)$, $t\in I=(\frac{B}{b},\infty)$, is a positive maximal solution of (1) on I by Lemma 2 and $\tilde{x}(1)=1$. Thus $x_+=\tilde{x}$ by Lemma 9, and consequently $x_+(t)=\frac{1}{b}y(bt)$ for $t\in(A_+,\infty)$ which yields $y=x_b$. Consequently, $\mathcal{A}_+\subset\mathcal{B}$. We have proved that $\mathcal{B}=\mathcal{A}_+$. □

THEOREM 3. *Let* $0<k<1$ *and set*

$$x_b(t)=bx_+\Big(\frac{t}{b}\Big) \quad \text{for } t\in(-\infty,bA_+),\ b<0. \tag{21}$$

Then $\mathcal{A}_-=\{x_b : b\in(-\infty,0)\}$.

Proof. From Lemmas 2 and 7 we deduce that z is a negative maximal solution of (1) on an interval J and the equation $z(t)-t=0$ has a (unique) solution if and only if $x(t)=-z(-t)$, $t\in I=\{t : -t\in J\}$, is a maximal positive solution of (1) on I and the equation $x(t)-t=0$ has a (unique) solution. Hence, for each $b\in(-\infty,0)$, the function x_b defined by (21) belongs to the set $\mathcal{A}_-$ and so the assertion of our theorem follows immediately from Theorem 2. □

REMARK 5. Theorems 2 and 3 and properties of x_+ given in Lemma 9 show that for each $c\in\mathbb{R}\setminus\{0\}$ there exists exactly one maximal solution x of (1) on an interval J such that $\lim_{t\in J,\,t\to c}x(t)=0$ and the equation $x(t)-t=0$ has a (unique) solution.

4. Equation $x(t)x'(t)=kx(x(t))$, $-1<k<0$. Let $-1<k<0$ and x be a nontrivial maximal solution of (1) on an interval J. By Theorem 1, $x(t)\neq 0$ and $x'(t)<0$ for $t\in J$. Hence either $x>0$ or $x<0$ on J.

LEMMA 10. *Let $-1 < k < 0$ and $x > 0$ be a maximal solution of* (1) *on an interval J. Then $J = [A, B]$, where $0 < A < B$ and $x(A) = B$, $x(B) = A$, $x' < 0$, $x'' < 0$ on $[A, B]$ and there exists a unique $a \in (A, B)$ such that $x(a) = a$.*

Proof. By Theorem 1, $x'(t) < 0$ for $t \in J$ and from the equality $(x')^2 + xx'' = kx'(x)x' < 0$ it follows: $x''(t) < 0$ for $t \in J$. If $\sup\{t : t \in J\} = \infty$ then $\lim_{t\to\infty} x(t) \geq 0$, contrary to $\lim_{t\to\infty} x'(t) < 0$ since x' is decreasing on J. Assume that $\inf\{t : t \in J\} = -\infty$. If $\lim_{t\to-\infty} x(t) = \infty$ then $J = \mathbb{R}$ by Lemma 1, contrary to $\sup\{t : t \in J\} < \infty$. Therefore $\lim_{t\to-\infty} x(t) = \alpha \in (0, \infty)$, and so $\lim_{t\to-\infty} x'(t) = 0$. From

$$0 = \lim_{t\to-\infty} x(t)x'(t) = k \lim_{t\to-\infty} x(x(t)) = k \lim_{t\to\alpha^-} x(t)$$

we see that $\lim_{t\to\alpha^-} x(t) = 0$. Since $\lim_{t\to-\infty} x(t) = \alpha$, $\lim_{t\to\alpha^-} x(t) = 0$, $(-\infty, \alpha) \subset J$ and x is decreasing on J, there exists a unique $t_0 \in (0, \alpha)$ such that $x(t_0) = t_0$. We now set

$$p(t) = \int_{x(t)}^{t} \frac{1}{x(s)}\, ds \quad \text{for } t \in J.$$

Then

$$p'(t) = \frac{1}{x(t)} - \frac{x'(t)}{x(x(t))} = \frac{1-k}{x(t)}, \quad t \in J$$

and integrating the equality $p'(t) = \frac{1-k}{x(t)}$ from t_0 to $t \in J$ it is easy to check that

$$\int_{x(t)}^{t_0} \frac{1}{x(s)}\, ds = -k \int_{t_0}^{t} \frac{1}{x(s)}\, ds \quad \text{for } t \in J. \tag{22}$$

Letting $t \to \alpha^-$ in (22) yields $\int_0^{t_0} \frac{1}{x(s)}\, ds = -k \int_{t_0}^{\alpha} \frac{1}{x(s)}\, ds$. Since $\int_0^{t_0} \frac{1}{x(s)}\, ds < \infty$, it follows that

$$\int_{t_0}^{\alpha} \frac{1}{x(s)}\, ds < \infty. \tag{23}$$

Letting $t \to -\infty$ in (22) we get $\int_\alpha^{t_0} \frac{1}{x(s)}\, ds = k \int_{-\infty}^{t_0} \frac{1}{x(s)}\, ds$. Since $\frac{1}{x(t)} > \frac{1}{\alpha}$ for $t \in J$, we see that $\int_{-\infty}^{t_0} \frac{1}{x(s)}\, ds = \infty$. Consequently, $\int_{t_0}^{\alpha} \frac{ds}{x(s)} = \infty$, contrary to (23). Hence $\inf\{t : t \in J\} > -\infty$. Set

$$\inf\{t : t \in J\} = A, \quad \sup\{t : t \in J\} = B.$$

Then $-\infty < A < B < \infty$. Assume that $J \neq [A, B]$. Let $\lim_{t\to A^+} x(t) = B_1$ and $\lim_{t\to B^-} x(t) = A_1$ (for $A \in J$ we have $B_1 = x(A)$ and for $B \in J$ we have $A_1 = x(B)$). Then $[A_1, B_1] \subset [A, B]$ and arguing as in the proof of Lemma 6 we can verify that the function $w : [A, B] \to [A, B]$,

$$w(t) = \begin{cases} B_1 & \text{for } t = A \\ x(t) & \text{for } t \in (A, B) \\ A_1 & \text{for } t = B \end{cases}$$

is a continuation of x, a contradiction. Whence $J = [A, B]$. Assume, on the contrary, $x(A) < B$ (the case $x(B) > A$ can be treated in a similar way). Consider the problem

$$z' = k\frac{x(z)}{z},$$

$$z(A) = x(A).$$

This problem has a unique solution z in a neighbourhood $\mathcal{U}$ of the point $t = A$ and the function

$$u(t) = \begin{cases} x(t) & \text{for } t \in J \\ z(t) & \text{for } t \in \mathcal{U} \setminus J \end{cases}$$

is a left continuation of x, a contradiction. We have proved that $x(A) = B$ and $x(B) = A$ and since $x > 0$ on J, we have $A = x(B) > 0$.

Assume that $x(t) \neq t$ for $t \in [A, B]$, for example let $x(t) > t$ for $t \in [A, B]$. Then

$$B = x(A) > A, \qquad A = x(B) > B,$$

which is impossible. Hence $x(a) = a$ for some $a \in (A, B)$ and the uniqueness of a follows from the fact that x is decreasing on $[A, B]$. □

LEMMA 11. *Let $-1 < k < 0$. Then there exists the unique maximal solution x of (1) on an interval $[a, b]$, $0 < a < 1 < b$, such that $x(1) = 1$.*

Proof. By Lemma 5, there exists the unique solution x_1 of (1) on an interval $[1-\varepsilon, 1+\varepsilon]$, $\varepsilon > 0$, such that $x_1(1) = 1$. Let x, y be maximal solutions of (1) which are the continuations of x_1. By Lemma 10, x and y are defined on intervals $[a_1, b_1]$ and $[a_2, b_2]$, respectively, $0 < a_1 < 1 < b_1$, $0 < a_2 < 1 < b_2$ and $x(a_1) = b_1$, $x(b_1) = a_1$, $y(a_2) = b_2$, $y(b_2) = a_2$, $x' < 0$ on $[a_1, b_1]$ and $y' < 0$ on $[a_2, b_2]$. Assume that $x \neq y$ and set

$$\alpha = \min\{t : x(s) = y(s) \text{ for } t \leq s \leq 1\},$$

$$\beta = \max\{t : x(s) = y(s) \text{ for } 1 \le s \le t\}$$

Then $\alpha \le 1 - \varepsilon$, $\beta \ge 1 + \varepsilon$ and from the equalities

$$\begin{aligned} 0 &= x(t)x'(t) - y(t)y'(t) = k\Big(x(x(t)) - y(y(t))\Big) \\ &= k\Big(x(x(t)) - x(y(t)) + x(y(t)) - y(y(t))\Big) \\ &= k\Big(x(y(t)) - y(y(t))\Big) \end{aligned}$$

for $t \in [\alpha, \beta]$, we deduce that $\alpha \le y(t) \le \beta$ for $t \in [\alpha, \beta]$. To prove that

$$x(\alpha) = y(\alpha) = \beta, \quad x(\beta) = y(\beta) = \alpha, \tag{24}$$

we assume for example that $x(\alpha) = y(\alpha) < \beta$ (the case $x(\beta) = y(\beta) > \alpha$ is treated similarly). Then $y(\alpha - \tau) \le \beta$, $x(\alpha - \tau) \le \beta$ with some $\tau > 0$ and for $t \in [\alpha - \tau, \alpha]$ we have

$$\begin{aligned} x(t)x'(t) - y(t)y'(t) &= k\Big(x(x(t)) - x(y(t)) + x(y(t)) - y(y(t))\Big) \\ &= k\Big(x(x(t)) - x(y(t))\Big) = kx'(\eta)(x(t) - y(t)) \end{aligned}$$

where η lies between $x(t)$ and $y(t)$. Therefore

$$(x^2(t) - y^2(t))' = \frac{2kx'(\eta)}{x(t) + y(t)}(x^2(t) - y^2(t)) \tag{25}$$

for $t \in [\alpha - \tau, \alpha]$. Set

$$K = \frac{2|k| \max\{|x'(t)| : a_1 \le t \le b_1\}}{a_1 + a_2},$$

$$R(t) = \max\{|x^2(s) - y^2(s)| : t \le s \le \alpha\} \qquad \text{for } t \in [\alpha - \tau, \alpha].$$

Then $R(\alpha) = 0$, $R > 0$ on $[\alpha - \tau, \alpha)$ and from (25) it follows that

$$|x^2(t) - y^2(t)| \le K \int_t^\alpha |x^2(s) - y^2(s)|\, ds \le KR(t)(\alpha - t)$$

and

$$R(t) \le KR(t)(\alpha - t)$$

for $t \in [\alpha - \tau, \alpha]$. Consequently, $1 \le K(\alpha - t)$ for $t \in [\alpha - \tau, \alpha)$, which is impossible. Therefore (24) is true.

Since $x \neq y$, by our assumption, and $x(\alpha) = y(\alpha) = \beta$, $y(\beta) = x(\beta) = \alpha$ by (24), we see that $a_i < \alpha$, $\beta < b_i$ for $i = 1, 2$. Set

$$M_-(t) = \max\{|x(s) - y(s)| : t \leq s \leq \alpha\} \quad \text{for } t \in [\max\{a_1, a_2\}, \alpha],$$

$$M_+(t) = \max\{|x(s) - y(s)| : \beta \leq s \leq t\} \quad \text{for } t \in [\beta, \min\{b_1, b_2\}].$$

Then $M_-(\alpha) = 0$, $M_-(t) > 0$ for $t \in [\max\{a_1, a_2\}, \alpha)$, $M_+(\beta) = 0$ and $M_+(t) > 0$ for $t \in (\beta, \min\{b_1, b_2\}]$. Since $x' < 0$ and $x'' < 0$ on $[a_1, b_1]$ by Lemma 10 and $x'(b_1) = k\frac{x(x(b_1))}{x(b_1)} = k\frac{b_1}{a_1}$, we have $|x'(t)| \leq |k|\frac{b_1}{a_1}$ for $t \in [a_1, b_1]$. In addition, $x(t) + y(t) \geq a_1 + a_2$ for $t \in [\max\{a_1, a_2\}, \min\{b_1, b_2\}]$. From the equalities

$$x^2(t) - y^2(t) = 2k \int_\beta^t (x(x(s)) - y(y(s)))\, ds,$$

$$x(x(t)) - y(y(t)) = x'(\varrho)(x(t) - y(t)) + x(y(t)) - y(y(t))$$

(here ϱ lies between $x(t)$ and $y(t)$), we deduce that (for $t \in [\beta, \min\{b_1, b_2\}]$)

$$\begin{aligned} & x(t) - y(t) \\ = \; & \frac{2k}{x(t) + y(t)} \int_\beta^t \Big(x'(\varrho)(x(s) - y(s)) + x(y(s)) - y(y(s))\Big)\, ds \end{aligned} \tag{26}$$

and

$$|x(t) - y(t)| \leq TM_+(t)(t - \beta) + L \int_\beta^t |x(y(s)) - y(y(s))|\, ds$$

where

$$T = \frac{2k^2 b_1}{a_1(a_1 + a_2)}, \quad L = \frac{2|k|}{a_1 + a_2}. \tag{27}$$

Then

$$M_+(t) \leq TM_+(t)(t - \beta) + L \int_\beta^t |x(y(s)) - y(y(s))|\, ds$$

and

$$M_+(t)(1 - T(t - \beta)) \leq L \int_\beta^t |x(y(s)) - y(y(s))|\, ds$$

for $t \in [\beta, \min\{b_1, b_2\}]$. Hence

$$|x(t) - y(t)| \le M_+(t) \le 2L \int_\beta^t |x(y(s)) - y(y(s))|\, ds \tag{28}$$

for $t \in [\beta, \min\{b_1, b_2, \beta + \frac{1}{2T}\}]$.

From the equality

$$x(t) - y(t) = \frac{2k}{x(t) + y(t)} \int_t^\alpha \Big(x'(\varrho)(x(s) - y(s)) + x(y(s)) - y(y(s))\Big)\, ds$$

which can be proved similarly to (26), we obtain

$$|x(t) - y(t)| \le T \int_t^\alpha |x(s) - y(s)|\, ds + L \int_t^\alpha |x(y(s)) - y(y(s))|\, ds \tag{29}$$

for $t \in [\max\{a_1, a_2\}, \alpha]$, where T and L are defined by (27). Since y is decreasing and $y(\alpha) = \beta$ by (24), there exists $c > 0$ such that $y(t) \in [\beta, \min\{b_1, b_2, \beta + \frac{1}{2T}\}]$ for $t \in [\alpha - c, \alpha]$, and consequently for these t we have (cf. (28) and (29))

$$\begin{aligned} &|x(t) - y(t)| \\ \le\ & T \int_t^\alpha |x(s) - y(s)|\, ds + 2L^2 \int_t^\alpha \int_\beta^{y(s)} |x(y(\tau)) - y(y(\tau))|\, d\tau\, ds. \end{aligned} \tag{30}$$

From $y(y(\alpha)) = \alpha$, $y'(\alpha) = k\frac{\alpha}{\beta}$, $y'(\beta) = k\frac{\beta}{\alpha}$ and

$$\Big(y(y(t))\Big)'_{t=\alpha} = y'(\beta) y'(\alpha) = k^2 < 1$$

we see that without loss of generality we can assume $y(y(t)) \ge t$ for $t \in [\alpha - c, \alpha]$. Then

$$\begin{aligned} &\int_t^\alpha \int_\beta^{y(s)} |x(y(\tau)) - y(y(\tau))|\, d\tau\, ds \\ =\ & \int_t^\alpha \int_{y(y(s))}^\alpha |x(\tau) - y(\tau)||(y^{-1}(\tau))'|\, d\tau\, ds \\ \le\ & m \int_t^\alpha \int_s^\alpha |x(\tau) - y(\tau)|\, d\tau\, ds \le \frac{m}{2}(\alpha - t)^2 M_-(t) \end{aligned} \tag{31}$$

for $t \in [\alpha - c, \alpha]$ where $m = \max\{|(y^{-1}(t))'| : a_2 \le t \le b_2\} > 0$. Now from (30) and (31) we obtain

$$\begin{aligned} |x(t) - y(t)| &\le T \int_t^\alpha |x(s) - y(s)|\, ds + L^2 m(\alpha - t)^2 M_-(t) \\ &\le (T + L^2 m(\alpha - t))(\alpha - t) M_-(t) \end{aligned}$$

for $t \in [\alpha - c, \alpha]$. Hence

$$M_-(t) \le (T + L^2 m(\alpha - t))(\alpha - t) M_-(t), \quad t \in [\alpha - c, \alpha]$$

and $1 \le (T + L^2 m(\alpha - t))(\alpha - t)$ for $t \in [\alpha - c, \alpha)$, which is impossible. We have proved that $\alpha = a_1 = a_2$ and $\beta = b_1 = b_2$, and so $x = y$. □

Let $-1 < k < 0$. By Lemmas 10 and 11, there exists the unique maximal solution x_- of (1) on an interval $[A, B]$ such that $x_-(1) = 1$. In addition, $0 < A < 1 < B$, $x_-(A) = B$, $x_-(B) = A$ and $x_-' < 0$, $x_-'' < 0$ on $[A, B]$. In the next two theorems we describe by x_- the set of all nontrivial maximal solutions x of (1) with $k \in (-1, 0)$

THEOREM 4. *Let $-1 < k < 0$ and set*

$$x_a(t) = a x_-\Big(\frac{t}{a}\Big) \quad \textit{for } t \in [aA, aB],\ a > 0. \tag{32}$$

Then $\mathcal{B}_+ = \{x_a : a \in (0, \infty)\}$ is the set of all positive maximal solutions of (1).

Proof. Let $a \in (0, \infty)$ and x_a be defined on $[aA, aB]$ by (32). Then x_a is a positive maximal solution of (1) on the interval $[aA, aB]$ by Lemma 2, and $x_a(a) = a x_-(1) = a$. Suppose that y is another positive maximal solution of (1) on an interval J such that $y(a) = a$. Then $J = [A_0, B_0]$, $0 < A_0 < a < B_0$, by Lemma 10, and the function $\tilde{x} : [\frac{A_0}{a}, \frac{B_0}{a}] \to \mathbb{R}$, $\tilde{x}(t) = \frac{1}{a} y(at)$ is a maximal solution of (1) on the interval $[\frac{A_0}{a}, \frac{B_0}{a}]$ by Lemma 2, and $\tilde{x}(1) = 1$. Thus $\tilde{x} = x_-$ by Lemma 11, and consequently $y = x_a$. Let z be a positive maximal solution of (1). Then $z(c) = c$ for a (unique) $c \in \mathbb{R}$ by Lemma 10, and from the above considerations we see that $x = x_c$. Hence $\mathcal{B}_+ = \{x_a : a \in (0, \infty)\}$ is the set of all maximal positive solutions of (1). □

THEOREM 5. *Let $-1 < k < 0$ and set*

$$x_b(t) = b x_-\Big(\frac{t}{b}\Big) \quad \textit{for } t \in [bB, bA],\ b < 0. \tag{33}$$

Then $\mathcal{B}_- = \{x_b : b \in (-\infty, 0)\}$ is the set of all negative maximal solutions of (1).

Proof. Let $b \in (-\infty, 0)$ and x_b be defined by (33). Then $x_b(b) = b$ and Lemma 2 implies that x_b is a negative maximal solution of (1) on the interval $[bB, bA]$. Arguing as in the proof of Theorem 4 we can verify that for any negative maximal solution y of (1) there exists $c \in (-\infty, 0)$ such that $y = x_c$.
□

REFERENCES

[1] G. Barba, Sopra l'equazione funzionale $f(x)f'(x) = f[f(x)]$ collegata ad un problema geometrico, *Atti Accad. Naz. Lincei Rend. C1. Sci. Fis. Mat. Natur.*, **11**(1930), 655-658.

[2] G. Barba, Sopra l'equazione funzionale $f(x)f'(x) = f[f(x)]$ collegata ad un problema geometrico II, *Atti Accad. Naz. Lincei Rend. C1. Sci. Fis. Mat. Natur.*, **11**(1930), 735-740.

[3] R.D. Driver, *Ordinary and Delay Differential Equations*, Springer, New York, 1977.

[4] G.M. Dunkel, On nested functional differential equations, *SIAM J. Appl. Math.*, **18**(1970), 514–525.

[5] G.M. Dunkel, Function differential equations: examples and problems, *Lectures Notes in Mathematics*, 144, Springer, 1970, 49–63.

[6] E. Eder, The functional differential equation $x'(t) = x(x(t))$, *J. Differential Equations*, **54**(1984), 390–400.

[7] M. Fečkan, On certain type of functional differential equations, *Math. Slovaca*, **43**(1993), 39–43.

[8] J.G. Si, W.R. Li and S.S. Cheng, Analytic solutions of an iterative functional differential equation, *Comput. Math. Appl.*, **33**(1997), 47–51.

[9] J.G. Si and S.S. Cheng, Analytic solutions of a functional differential equation with state dependent argument, *Taiwanese J. Math.*, **1**(1997), 471–480.

[10] J.G. Si and X.P. Wang, Smooth solutions of nonhomogeneous iterative functional differential equation with variable coefficients, *J. Math. Anal. Appl.*, **226**(1998), 377–392.

[11] J.G. Si, X.P. Wang and S.S. Cheng, Analytic solutions of a functional differential equations with a state derivative dependent delay, *Aequationes Math.*, **57**(1999), 75–86.

[12] J.G. Si, X.P. Wang and S.S. Cheng, Nondecreasing and convex C^2-solutions of an iterative functional-differential equation, *Aequationes Math.*, **60**(2000), 38–56.

[13] S. Staněk, On global properties of solutions of functional differential equation $x'(t) = x(x(t)) + x(t)$, *Dynamical Systems and Appl.*, 4(1995), 263-278.

[14] S. Staněk, Global properties of decreasing solutions of the equation $x'(t) = x(x(t)) + x(t)$, *Functional Differential Equations*, 4(1997), 191–213.

[15] S. Staněk, Global properties of solutions of iterative-differential equations, *Functional Differential Equations*, 5(1998), 463–481.

[16] S. Staněk, Global properties of increasing solutions for the equation $x'(t) = x(x(t)) - bx(t)$, $b \in (0,1)$, *Soochow Journal of Mathematics*, **26**(2000), 37–65.

[17] S. Staněk, Global properties of decreasing solutions for the equation $x'(t) = x(x(t)) - bx(t)$, $b \in (0,1)$, *Soochow Journal of Mathematics*, **26**(2000), 123–134.

[18] S. Staněk, On global properties of solutions of the equation $y'(t) = ay(t - by(t))$, *Hokkaido Math. J.*, **30**(2001), 75–89.

[19] W. Ke, On the equation $x'(t) = f(x(x(t)))$, *Funkcialaj Ekvacioj*, **33**(1990), 405–425.