STABILITY AND LINEARIZATION FOR DIFFERENTIAL EQUATIONS WITH A DISTRIBUTED DELAY*

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Dedicated to the memory of Anatolii Dmitrievich Myshkis

Abstract. We present some new stability results for nonlinear equations with a distributed delay

\[ \dot{x}(t) + \sum_{k=1}^{m} \int_{h_k(t)}^{t} f_k(s, x(s)) d_s R_k(t, s) = 0. \]

The results are applied to establish local stability of some models of population dynamics: the Nicholson’s blowflies equation and the Mackey-Glass equation describing white blood cells production.

Key Words. Distributed delay, stability, linearization, Nicholson’s blowflies equation, Mackey-Glass equation

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1. Introduction. Equations with a distributed delay provide a more flexible and realistic description for real world phenomena than ordinary differential equations or equations with concentrated delays [1]. If a maturation delay is incorporated in the equation, then the maturation time is, generally, not constant, but is distributed around its expectancy value. The same is valid for the digestion delay, as well as for the latency period for most infectious diseases.

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Historically, equations with a distributed delay were studied even before relevant models with concentrated delays appeared. To the best of our knowledge the first systematic study of equations with a distributed delay can be found in the monograph of Myshkis [2], the results obtained by 1993 are summarized in the book of Kuang [1]. Presently equations with a distributed delay are intensively studied. For various models of mathematical biology with distributed and concentrated delays see the monographs [1, 3, 4, 5]. Distributed delays mainly arise when modeling real world phenomena, such as driver's behaviour, optic systems, tumor-immune system interaction, infectious diseases treatment, see, for example, recent papers [6]-[9] and references therein. Here we do not mention extensive literature on neural networks and control theory for equations with distributed delays, as well as partial differential equations incorporating this type of delays.

In most publications integrodifferential equations are studied, see, for example, [10]; however sometimes applied models are assumed to incorporate both integral terms and equations with concentrated delays (see, for example, [11, 12]); such equations are considered in the present paper. Most of previously obtained results are not relevant for non-autonomous models and do not involve equations with a concentrated delay as a special case. The present paper fills up this gap.

The paper is organized as follows. In Section 2 we present stability results for linear equations. Theorem 1 is a new result, and the Mean Value Theorem for the equation with distributed delays (Lemma 3) is for the first time formulated in the form accounting for each delay. Further, we apply these results to nonlinear equations. Section 3 is concerned with the theoretical justification of the linearization process presented in Theorem 2 and its corollary. There are many results of this type for nonlinear delay differential equations. However to the best of our knowledge, none of the linearized stability theorems can be applied to nonlinear differential equations with distributed delays and measurable parameters which are considered in this paper. Finally, in Section 4 local stability of the unique positive equilibrium is studied for some applied models, such as the Nicholson’s blowflies equation and the Mackey-Glass equation describing the production of white blood cells.

2. Preliminaries and Results for Linear Equations. We consider linear and nonlinear scalar differential equations with a distributed delay

\[ \dot{x}(t) + \sum_{k=1}^{m} \int_{h_k(t)}^{t} x(s) d_k R_k(t, s) = 0, \quad t \geq t_0 \geq 0, \]
\( \dot{x}(t) + \sum_{k=1}^{m} \int_{h_k(t)}^{t} x(s) d_s R_k(t, s) + \sum_{j=1}^{q} \int_{g_j(t)}^{t} f_j(s, x(s)) d_s G_j(t, s) = 0, \ t \geq t_0 \geq 0, \)

as well as both equations with a nondelay term, which for (1) becomes

\( \dot{x}(t) + b(t)x(t) + \sum_{k=1}^{m} \int_{h_k(t)}^{t} x(s) d_s R_k(t, s) = 0, \ t \geq t_0 \geq 0. \)

Here we assume that for each \( t \) the memory is finite and define

\( h_k(t) = \inf \{ s \leq t | R_k(t, s) \neq 0 \}, \ g_j(t) = \inf \{ s \leq t | G_j(t, s) \neq 0 \}. \)

We consider equations (1), (2) and (3) for a fixed \( t_0 \geq 0 \) with the initial condition

\( x(t) = \varphi(t), \ t \leq t_0 \)

under the following assumptions:

(a1) \( R_k(t, \cdot), G_j(t, \cdot) \) are left continuous functions of bounded variation for any \( t \); \( R_k(\cdot, s), G_j(\cdot, s) \) are locally integrable for any \( s \), \( R_k(t, h_k(t)) = 0 \), \( G_j(t, g_j(t)) = 0 \) and \( R_k(t, s), G_j(t, s) \) are constant for \( s > t \) and coincide with the right limits \( R_k(t, t^+), G_j(t, t^+) \); the functions

\( \alpha_k(t) = \bigvee_{\tau \in [h_i, t^+]} R_k(t, \tau), \ \beta_j(t) = \bigvee_{\tau \in [g_j, t^+]} G_j(t, \tau) \)

are Lebesgue measurable and bounded on \( [0, \infty) \), \( k = 1, \ldots, m \), \( j = 1, \ldots, q \), where \( \bigvee_{\tau \in I} f(\tau) \) denotes the variation of the function \( f \) on segment \( I \);

(a2) \( h_k, g_j : [0, \infty) \rightarrow \mathbb{R}, \ k = 1, \ldots, m, \ j = 1, \ldots, q \) are Lebesgue measurable functions, \( h_k(t) \leq t, \ g_j(t) \leq t, \)

\( \limsup_{t \to \infty} [t - h_k(t)] < \infty, \ \limsup_{t \to \infty} [t - g_j(t)] < \infty. \)

The integral \( \int_{h_k(t)}^{t} x(s) d_s R_k(t, s) \) is understood as \( \int_{h_k(t)}^{t^+} x(s) d_s R_k(t, s) \), we have

\( \int_{h_k(t)}^{t} x(s) d_s R_k(t, s) = \int_{h_k(t)}^{t+\varepsilon} x(s) d_s R_k(t, s) \)

for any \( \varepsilon > 0. \)
For some particular choices of $R_k(t, s)$ we obtain the following equations as special cases of (1):

(7) $\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) = 0,$

(8) $\dot{x}(t) + \int_{h(t)}^{t} K(t, s)x(s) \, ds = 0,$

(9) $\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) + \int_{h(t)}^{t} K(t, s)x(s) \, ds = 0,$

where

(10) $R_k(t, s) = a_k(t)\chi(h_k(t),\infty)(s),$

$\chi_I$ is the characteristic function of interval $I$,

(11) $R(t, s) = \int_{h(s)}^{s} K(t, \tau) \, d\tau,$

and

(12) $R_k(t, s) = a_k(t)\chi(h_k(t),\infty)(s), \ k = 1, \cdots, m, \quad R_{m+1}(t, s) = \int_{h(s)}^{s} K(t, \tau) \, d\tau,$

respectively. Here $K(t, s) = 0, \ s \not\in [h(t), t], \ \int_{h(t)}^{t} |K(t, s)| \, ds$ is a Lebesgue measurable function bounded on $[0, \infty)$, $a_k(t)$ are Lebesgue measurable functions bounded on $[0, \infty)$, $k = 1, \cdots, m$. Then the relevant functions $R_k(t, s)$ for (7)-(9) satisfy (a1); we also assume that (a2) holds for $h(t), \ h_k(t), \ k = 1, \cdots, m$.

Now let us proceed to the initial function $\varphi$. This function should satisfy such conditions that the integral in the left hand side of (1) exists almost everywhere. In particular, if $R(t, \cdot)$ is absolutely continuous for any $t$ (which allows us to write (1) as an integrodifferential equation), then $\varphi$ can be chosen as a Lebesgue measurable essentially bounded function. If $R_k(t, \cdot)$ is a combination of step functions (which corresponds to an equation with concentrated delays) then $\varphi$ should be a Borel measurable bounded function. For any choice of $R$ the integral exists if $\varphi$ is bounded and continuous. Thus, we assume
(a3) $\phi : (-\infty, 0] \to \mathbb{R}$ is a bounded continuous function; the following hypothesis should be satisfied for functions $f_j$ in (2):

(a4) $f_j(t,u)$ are Lebesgue measurable essentially bounded in $[0,\infty)$ functions of the first argument and are continuous and bounded in the second argument, $f_j(t,0) = 0$, $j = 1, \cdots, q$.

Everywhere below we will assume that for all equations and initial conditions hypotheses (a1)-(a4) hold.

**Definition.** An absolutely continuous function $x : \mathbb{R} \to \mathbb{R}$ is called a solution of the problem (2),(5) if it satisfies equation (2) for almost all $t \in [t_0, \infty)$ and conditions (5) for $t \leq t_0$.

In addition to linear equation (1) we will also consider the following non-homogeneous equation

$$\dot{x}(t) + \sum_{k=1}^{m} \int_{h_k(t)}^{t} x(s) \, d_s R_k(t,s) = f(t),$$

where $f(t)$ is a Lebesgue measurable locally essentially bounded function.

**Definition.** For each $s \geq t_0$ and $t \geq s$ the solution $X(t,s)$ of the problem

$$\dot{x}(t) + \sum_{k=1}^{m} \int_{h_k(t)}^{t} x(\tau) \, d_\tau R_k(t,\tau) = 0, \quad t \geq s,$$

$x(t) = 0$, $t < s$, $x(s) = 1$,

is called the fundamental function of equation (13). Here $X(t,s) = 0$, $0 \leq t < s$.

**Lemma 1.** [13, 14] The solution of the initial value problem (13),(5) can be presented as

$$x(t) = \int_{t_0}^{t} X(t,s) \varphi(t_0) \, - \int_{t_0}^{t} X(t,s) \left[ \sum_{k=1}^{m} \int_{h_k(s)}^{s} \varphi(\zeta) d_\zeta R_k(s,\zeta) \right] \, ds$$

$$+ \int_{t_0}^{t} X(t,s) f(s) \, ds,$$

where $\varphi(t) = 0$, $t > t_0$.

**Definition.** Equation (1) is (uniformly) exponentially stable, if there exist $K > 0$ and $\lambda > 0$ such that the fundamental function $X(t,s)$ defined by (14) has the estimate

$$|X(t,s)| \leq K \, e^{-\lambda(t-s)} , \quad t \geq s \geq 0.$$
Denote similar to (6)

\[ a_k(t) = R_k(t, t^+) \quad \alpha_k(t) = \bigvee_{\tau \in [h_k(t), t^+]} R_k(t, \tau) = \int_{h_k(t)}^t |d_\tau R_k(t, \tau)|. \]

For (7) functions \( a_k(t) \) coincide with the coefficients, \( \alpha_k(t) = |a_k(t)| \); in (8) we have

\[ a(t) = \int_{h(t)}^t K(t, s) \, ds, \quad \alpha(t) = \int_{h(t)}^t |K(t, s)| \, ds. \]

Let us introduce some functional spaces on a halfline. Denote by \( L^\infty_{[t_0, \infty)} \) the space of all essentially bounded on \([t_0, \infty)\) functions with the essential supremum norm

\[ \|y\|_{L^\infty} = \text{ess sup}_{t \geq t_0} |y(t)|, \]

by \( C_{[t_0, \infty)} \) the space of all continuous bounded on \([t_0, \infty)\) functions with the sup-norm.

**Lemma 2.** [15, 16] Suppose that for any \( f \in L^\infty_{[t_0, \infty)} \) the solution of (13) with the zero initial conditions belongs to \( C_{[t_0, \infty)} \). Then (1) is exponentially stable.

**Lemma 3.** Suppose \( R_k(t, \cdot) \) are nondecreasing functions. Then for any solution \( x(t) \) of (13) there exist \( g_k(t) \) such that

\[ h_k(t) \leq g_k(t) \leq t \]

and \( x(t) \) is also a solution of the equation

\[ \dot{x}(t) + \sum_{k=1}^m a_k(t)x(g_k(t)) = f(t), \]

where \( a_k(t) = R_k(t, t^+) = \alpha_k(t) \) was introduced in (17).

**Proof.** By the Mean Value Theorem for the Lebesgue Stiltjes integral, for any \( k = 1, \ldots, m \) and \( t \geq t_0 \) there exists \( g_k(t) \in [h_k(t), t] \) such that for the continuous function \( x(t) \) we have

\[ \int_{h_k(t)}^t x(s) \, d_s R_k(t, s) = x(g_k(t)) \int_{h_k(t)}^t d_s R_k(t, s) = a_k(t)x(g_k(t)). \]

Thus \( x(t) \) satisfying (13) is also a solution of (20).
Lemma 4. [17] Suppose
\[ a_k(t) \geq a_0 > 0, \quad \sup_{t \geq t_0} a_k(t) < \infty, \quad \sup_{t \geq t_0} (t - h_k(t)) < \infty, \quad k = 1, 2, \ldots, \]
and
\[ \limsup_{t \to \infty} \sum_{k=1}^{m} \frac{a_k(t)}{a_i(t)} \int_{h_k(t)}^{t} \sum_{i=1}^{m} a_i(s) ds < 1 + \frac{1}{e}. \]
Then equation (7) is exponentially stable.

Theorem 1. Suppose \( R_k(t, \cdot), k = 1, \ldots, m, \) are nondecreasing functions, \( a_k(t) \geq a_0 > 0 \) and condition (21) holds, where \( a_k(t) = R_k(t, t^+). \) Then equation (1) is exponentially stable.

Proof. Suppose \( x(t) \) is a solution of equation (13) with the zero initial conditions. By Lemma 3 there exist functions \( g_k(t), \) where \( h_k(t) \leq g_k(t) \leq t, \) such that \( x(t) \) also satisfies the linear equation with variable concentrated delays
\[ \dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(g_k(t)) = f(t). \]
Since \( h_k(t) \leq g_k(t) \) then \( \sup_{t \geq t_0} (t - g_k(t)) < \infty \) and condition (21) also holds if \( h_k(t) \) are replaced by \( g_k(t). \) Then by Lemma 4 equation (22) is exponentially stable.

For \( f \in L_{\infty}[t_0, \infty) \) the solution \( y(t) \) of (22) with the zero initial conditions has the form (15):
\[ y(t) = \int_{t_0}^{t} Y(t, s)f(s) ds, \]
where \( Y(t, s) \) is the fundamental function of equation (22). Since \( Y(t, s) \) has an exponential estimate in the form (16), then the solution \( y(t) \) of equation (22) belongs to \( C[t_0, \infty). \) It means that the same is true for equation (13). By Lemma 2 this equation is exponentially stable.

In Theorem 1 we suppose that \( R_k(t, \cdot) \) are nondecreasing functions. In the following results we can omit this restriction.

Lemma 5. [18] Suppose that there exist a set of indices \( I \subset \{1, 2, \ldots, l\} \) and numbers \( \beta > 0, \quad 1 > \gamma > 0, \) such that
\[ \sum_{k \in I} a_k(t) \geq \beta > 0, \]
\[ \sum_{k \in I} \int_{h_k(t)}^{t} \left( \int_{s}^{t} \sum_{j=1}^{m} \alpha_j(\tau) d\tau \right) |d_s R_k(t, s)| + \sum_{k \notin I} \alpha_k(t) \leq \gamma \sum_{k \in I} a_k(t) \]
for sufficiently large \( t > 0. \) Then equation (1) is exponentially stable.
Consider the equation with one delayed and one non-delayed terms

\[
\dot{x}(t) + b(t)x(t) + \int_{h(t)}^{t} x(s) d_R(t, s) = 0.
\]

Choosing \( I = \{1\} \) and \( I = \{1, 2\} \) and applying Lemma 5 we obtain the following result.

**Corollary 1.** [18] Suppose there exist \( \beta > 0 \) and \( \gamma \in (0, 1) \) such that at least one of the following conditions holds:

1) \( b(t) \geq \beta > 0, \quad \int_{h(t)}^{t} R(t, \tau) \leq \gamma b(t); \)
2) \( R(t, t^+) + b(t) \geq \beta > 0, \)

\[
|b(t)| \int_{h(t)}^{t} \left( \int_{s}^{t} R(s, \tau) + |b(\tau)| \right) d\tau \left| d_R(t, s) \right| \leq \gamma [R(t, t^+) + b(t)].
\]

Then equation (25) is exponentially stable.

Now consider the equation with a non-delay term and proportional coefficients

\[
\dot{x}(t) + r(t) \left( Bx(t) + \int_{h(t)}^{t} x(s) d_R(t, s) \right) = 0.
\]

**Corollary 2.** [18] Let \( r(t) \geq r_0 > 0 \) and \( R(t, \cdot) \) be a nondecreasing function satisfying \( R(t, t^+) = \int_{h(t)}^{t} d_R(t, s) = 1 \). Suppose that at least one of the following conditions holds:

1) \( B > 0, |A| < B; \)
2) \( A + B > 0, \quad |A| \int_{h(t)}^{t} \left( \int_{s}^{t} r(\tau) d\tau \right) d_R(t, s) < \frac{A + B}{|A| + |B|}. \)

Then (26) is exponentially stable.

Consider the function

\[
\omega(\sigma) = \int_{0}^{\infty} |U_\sigma(s)| ds, \quad 0 < \sigma < \frac{\pi}{2},
\]

where \( U_\sigma(t) \) is the solution of the following initial value problem for the autonomous delay equation

\[
\dot{x}(t) + x(t - \sigma) = 0, \quad x(t) = 0, \quad t < 0, \quad x(0) = 1.
\]

In [19, 20] properties of \( \omega(\sigma) \) were obtained and the values of \( \omega(\sigma) \) were tabulated. In particular, it was shown that the constant

\[
U = \sup_{\sigma \omega(\sigma) < 1} \left( \sigma + \frac{1}{\omega(\sigma)} \right)
\]
is approximately

\[ U \approx 1.425. \]

**Lemma 6.** [18] Suppose that \( R_k(t, \cdot) \) are nondecreasing functions, \( \sum_{k=1}^{m} R_k(t, t^+) \neq 0 \) almost everywhere and

\[ a(t) := \sum_{k=1}^{m} R_k(t, t^+) \geq 0, \quad \int_{0}^{\infty} \sum_{k=1}^{m} R_k(t, t^+) dt = \infty, \]

\[ \limsup_{t \to \infty} \int_{\min h_k(t)}^{t} a(s) ds < U \approx 1.425, \]

Then (1) is exponentially stable.

3. Stability by the First Approximation. In order to introduce stability by the first approximation, we consider the initial value problem (2) with an arbitrary initial point and initial conditions (5).

We assume that (a1)-(a4) hold, which means that the initial function is continuous. However for some particular cases (for example, the equation with several concentrated delays and the integrodifferential equation) we can omit this restriction and assume that \( \varphi \) is a Borel measurable bounded function.

We present the following local stability definitions (see [13]).

**Definition.** We will say that the zero solution of (2) is (locally) uniformly stable if for any \( \varepsilon > 0 \) and \( t_0 \geq 0 \) there exists \( \delta > 0 \) such that for any initial conditions \( |\varphi(t)| < \delta, \ t \leq t_0 \), for the solution \( x(t) \) of (2)-(5) we have \( |x(t)| < \varepsilon, \ t \geq t_0 \), and the number \( \delta \) does not depend on initial point \( t_0 \).

The zero solution of equation (2) is (locally) uniformly asymptotically stable, if it is uniformly stable and there exists \( \delta > 0 \) such that for every \( \eta > 0 \), there is a \( t_1(\eta) \) such that \( |\varphi(t)| < \delta \) for \( t \leq t_0 \) implies \( |x(t)| < \eta \) for \( t \geq t_0 + t_1(\eta) \).

**Theorem 2.** Suppose that for any sufficiently small \( A > 0 \) there exists \( B > 0 \), where \( \lim_{A \to 0} B(A)/A = 0 \), such that inequality \( |u| < A \) implies \( |f_j(t, u)| < B, \ j = 1, \cdots, q, \ t \geq t_0 \).

In addition, suppose that the linear equation (1) is exponentially stable with estimation (16) for its fundamental function \( X(t, s) \). Then the zero solution of equation (2) is locally uniformly asymptotically stable.
Proof. Denote by \( x(t) \) the solution of (2) with initial conditions (5), by \( y(t) \) the solution of linear equation (1) with the same initial conditions,

\[
m_\varphi = \sup_{t \leq t_0} |\varphi(t)|, \quad a = \sup_{t \geq 0} \sum_{k=1}^{m} \int_{h_k(t)}^{t} |d_s R_k(t, s)|, \quad r = \sup_{t \geq 0} \sum_{j=1}^{q} \int_{g_j(t)}^{t} |d_s G_j(t, s)|, \quad H = \max_{k,j} \left\{ \sup_{t \geq t_0} (t-h_k(t)), \sup_{t \geq t_0} (t-g_j(t)) \right\}.
\]

Solution representation formula (15) implies

\[
y(t) = X(t, t_0)\varphi(t_0) - \int_{t_0}^{t} X(t, s) \left[ \sum_{k=1}^{m} \int_{h_k(s)}^{s} \varphi(\tau) d_\tau R_k(s, \tau) \right] ds,
\]

where \( \varphi(s) = 0 \), if \( s > t_0 \). Hence \( \varphi(\tau) = 0 \), \( \tau \in [h_k(t), t] \) for \( t > t_0 + H \) and

\[
|y(t)| \leq |X(t, t_0)||\varphi(t_0)| + \int_{t_0}^{t} |X(t, s)| \left[ \sum_{k=1}^{m} \int_{h_k(s)}^{s} |\varphi(\tau)| |d_\tau R_k(s, \tau)| \right] ds
\]

\[
\leq K e^{-\lambda(t-t_0)} m_\varphi + Ka m_\varphi \int_{t_0}^{t_0+H} e^{-\lambda(t-s)} ds \leq Mm_\varphi e^{-\lambda(t-t_0)},
\]

where \( M = K \left( 1 + \frac{a(e^{\lambda H} - 1)}{\lambda} \right) \).

Thus for the solution of (1),(5) we have

\[
|y(t)| \leq Mm_\varphi e^{-\lambda(t-t_0)}, \quad t \geq t_0,
\]

where the constant \( M \) does not depend on the initial data function \( \varphi(t) \). Without loss of generality we can assume \( M > 1 \).

By the assumptions of the theorem, \( f_j \) is continuous in \( u \) and

\[
\lim_{u \to 0} \left| \frac{f_j(t, u)}{u} \right| = 0
\]

for any \( j \) uniformly on \( t \in [t_0, \infty) \), so there exists \( \delta > 0 \) such that for any \( t \geq t_0, u \neq 0 \) the inequality \( |u| < \delta \) implies

\[
|f_j(t, u)| < \frac{\delta \lambda}{2M(r+1)}, \quad |f_j(t, u)| < \mu |u|,
\]

where \( \mu := \frac{\lambda e^{-\lambda H/2}}{4rK}, j = 1, \ldots, q. \)
First, let us prove that as far as

$$m_\varphi < \frac{\delta}{2M},$$

we have \(|x(t)| < \delta\) for any \(t\). In fact, assuming there are points where \(|x(t)| = \delta\) and taking \(\bar{t}\) which is the minimal among all such points we obtain by solution representation formula (15) and later applying (31), (32) and (16):

\[
|x(\bar{t})| = \left| y(\bar{t}) - \int_{t_0}^{\bar{t}} X(\bar{t}, s) \sum_{j=1}^q \left[ \int_{g_j(s)} f_j(\tau, x(\tau)) d\tau G_j(s, \tau) \right] ds \right| \\
\leq |y(\bar{t})| + \int_{t_0}^{\bar{t}} |X(\bar{t}, s)| \sum_{j=1}^q \left[ \int_{g_j(s)} |f_j(\tau, x(\tau))| d\tau G_j(s, \tau) \right] ds \\
\leq \frac{M\delta}{2M} + \frac{K\delta}{2M(r+1)} \int_{t_0}^{\bar{t}} \lambda e^{-\lambda(\bar{t}-t_0)} \, ds \\
< \frac{\delta}{2} + \frac{\delta}{2} = \delta,
\]

since \(K < M\); the contradiction obtained proves the fact that \(|x(t)| < \delta\) for any \(t\).

Further, we will prove that for the solution \(x(t)\) of problem (2),(5) the following estimation holds: if \(m_\varphi > 0\), then

$$|x(t)| < 2Mm_\varphi e^{-\lambda(t-t_0)/2}, \quad t \geq t_0.$$  

This inequality is obviously satisfied for \(t = t_0\) and in \([t_0, t_0 + \varepsilon]\) for some \(\varepsilon > 0\). Assume the contrary; let \(\bar{t}\) be the first point where this inequality becomes an equality. By (31) and (32)

\[
|x(\bar{t})| \leq |y(\bar{t})| + \int_{t_0}^{\bar{t}} X(\bar{t}, s) \sum_{j=1}^q \left[ \int_{g_j(s)} f_j(\tau, x(\tau)) d\tau G_j(s, \tau) \right] ds \\
\leq Mm_\varphi e^{-\lambda(\bar{t}-t_0)} + r \int_{t_0}^{\bar{t}} Ke^{-\lambda(\bar{t}-s)} \mu \max_{\tau \in [s-H,s]} |x(\tau)| ds \\
< Mm_\varphi e^{-\lambda(\bar{t}-t_0)/2} + rK\mu \int_{t_0}^{\bar{t}} e^{-\lambda(\bar{t}-s)} m_\varphi 2Me^{-\lambda(s-H-t_0)/2} ds \\
= \frac{4rKM}{\lambda} m_\varphi e^{\lambda H/2} m_\varphi e^{-\lambda(t_0)/2} \int_{t_0}^{\bar{t}} \frac{\lambda}{2} e^{-\lambda(t-s)/2} ds \\
< \frac{4rKe^{\lambda H/2}}{\lambda} Mm_\varphi e^{-\lambda(t_0)/2} \int_{-\infty}^{\bar{t}} \frac{\lambda}{2} e^{-\lambda(t-s)/2} ds \\
= 2Mm_\varphi e^{-\lambda(t_0)/2},
\]
the contradiction proves (34).

Since $M$ and $\mu$ do not depend on the initial conditions, as far as they do not exceed the value defined in (33), inequality (34) implies local uniform asymptotic stability of equation (2), which completes the proof.

Consider the nonlinear equation

\begin{equation}
\dot{x}(t) + \sum_{j=1}^{q} A_j(t) \int_{g_j(t)}^{t} f_j(s, x(s)) d_s G_j(t, s) = 0, \ t \geq 0,
\end{equation}

where $f_k(t, u)$ are differentiable in $u$ in some neighbourhood of $u = 0$, $f_k(t, 0) = 0$. Then the following corollary is valid.

**Corollary 3.** If $f_j(t, u)$ are differentiable in $u$, $j = 1, \cdots, q$ and the linear equation

\begin{equation}
\dot{y}(t) + \sum_{j=1}^{q} A_j(t) \int_{g_j(t)}^{t} \frac{\partial f_j}{\partial u}(s, 0) y(s) d_s G_j(t, s) = 0
\end{equation}

is exponentially stable, then the zero solution of (35) is locally uniformly asymptotically stable.

**4. Applications to Equations of Population Dynamics.** In this section we apply the previous results to equations of mathematical biology.

First we consider the Nicholson’s blowflies equation. Its version with a constant delay

\begin{equation}
\frac{dN}{dt} = PN(t-\tau)e^{-aN(t-\tau)} - \delta N(t)
\end{equation}

was introduced by Nicholson [21] to model the laboratory fly population; its dynamics was later investigated in [22] and [23]. We consider the Nicholson’s blowflies equation with a distributed delay

\begin{equation}
\frac{dN}{dt} = r(t) \left[ P \int_{h(t)}^{t} N(s)e^{-N(s)} d_s R(t, s) - \delta N(t) \right],
\end{equation}

where $R(t, \cdot)$ is a nondecreasing function satisfying $R(t, t^+) = 1$ (the delay distribution function has a probabilistic meaning), $P > 0$, $\delta > 0$, $r(t) \geq 0$, $\sup_{t \geq t_0}(t - h(t)) < \infty$.

Some aspects of global stability and oscillation of (37) were studied in [24], see also the recent review [25].

We can also consider (37) where the coefficients are not necessarily proportional

\begin{equation}
\frac{dN}{dt} = P(t) \int_{h(t)}^{t} N(s)e^{-N(s)} d_s R(t, s) - \delta(t)N(t).
\end{equation}
Then Corollary 1, Part 1 and Corollary 3 imply the following result.

**Theorem 3.** If there exists $\beta > 0, \gamma \in (0, 1)$ such that $\delta(t) \geq \beta$, $P(t) \leq \gamma \delta(t)$, then the zero solution of (38) is locally uniformly asymptotically stable.

If $P > \delta$, then in addition to the zero equilibrium, equation (37) also has a positive equilibrium $N^* = \ln(P/\delta)$. After the substitution $x(t) = N - N^*$ equation (37) takes the form

$$
\frac{dx}{dt} = \delta r(t) \left[ P \int_{h(t)}^{t} (x(s) + N^*) e^{-\gamma(s)} d_s R(t, s) - N^* - x(t) \right],
$$

its linearized version is

$$
\frac{dy}{dt} = -\delta r(t) \left[ \left( \ln \frac{P}{\delta} - 1 \right) \int_{h(t)}^{t} y(s) d_s R(t, s) + y(t) \right].
$$

Applying Corollary 2, Corollary 3, Lemma 6 and Theorem 1 we obtain local stability results.

**Theorem 4.** Suppose there exists $\alpha > 0$ such that $r(t) \geq \alpha > 0$ and at least one of the following conditions holds:

1) $1 < \frac{P}{\delta} < e^2$;

2) $\frac{P}{\delta} > e, \delta \left( \ln \frac{P}{\delta} - 1 \right) \int_{h(t)}^{t} r(t) ds R(t, s) < 1$;

3) $\frac{P}{\delta} > e, \ln \frac{P}{\delta} \limsup_{t \to \infty} \int_{h(t)}^{t} r(s) ds < U \approx 1.425$;

4) $\frac{P}{\delta} > e, \left( \ln \frac{P}{\delta} - 1 \right) \limsup_{t \to \infty} \int_{h(t)}^{t} r(s) ds < 1 + \frac{1}{e}$.

Then the positive equilibrium of equation (37) is locally uniformly asymptotically stable.

Evidently conditions 1) and 2) are independent of 3) and 4) (the first condition is delay-independent, the second test only depends on the distribution of $R(t, s)$), while 4) is sharper than 3) whenever

$$
\limsup_{t \to \infty} \int_{h(t)}^{t} r(s) ds > U - 1 - \frac{1}{e} \approx 0.057.
$$

Next, consider the Mackey-Glass Equation with variable coefficients and a distributed delay:

$$
\dot{N}(t) = A(t) \int_{h(t)}^{t} \frac{N(s)}{1 + [N(s)]^\gamma} d_s R(t, s) - B(t)N(t), \quad t \geq 0,
$$

$$
\frac{dx}{dt} = \delta r(t) \left[ P \int_{h(t)}^{t} (x(s) + N^*) e^{-\gamma(s)} d_s R(t, s) - N^* - x(t) \right],
$$

its linearized version is

$$
\frac{dy}{dt} = -\delta r(t) \left[ \left( \ln \frac{P}{\delta} - 1 \right) \int_{h(t)}^{t} y(s) d_s R(t, s) + y(t) \right].
$$

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**Theorem 4.** Suppose there exists $\alpha > 0$ such that $r(t) \geq \alpha > 0$ and at least one of the following conditions holds:

1) $1 < \frac{P}{\delta} < e^2$;

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3) $\frac{P}{\delta} > e, \ln \frac{P}{\delta} \limsup_{t \to \infty} \int_{h(t)}^{t} r(s) ds < U \approx 1.425$;

4) $\frac{P}{\delta} > e, \left( \ln \frac{P}{\delta} - 1 \right) \limsup_{t \to \infty} \int_{h(t)}^{t} r(s) ds < 1 + \frac{1}{e}$.

Then the positive equilibrium of equation (37) is locally uniformly asymptotically stable.

Evidently conditions 1) and 2) are independent of 3) and 4) (the first condition is delay-independent, the second test only depends on the distribution of $R(t, s)$), while 4) is sharper than 3) whenever

$$
\limsup_{t \to \infty} \int_{h(t)}^{t} r(s) ds > U - 1 - \frac{1}{e} \approx 0.057.
$$

Next, consider the Mackey-Glass Equation with variable coefficients and a distributed delay:

$$
\dot{N}(t) = A(t) \int_{h(t)}^{t} \frac{N(s)}{1 + [N(s)]^\gamma} d_s R(t, s) - B(t)N(t), \quad t \geq 0,
$$
where $R(t, \cdot)$ is a nondecreasing function satisfying $R(t, t^+) = 1$, $A(t) \geq 0$, $B(t) \geq 0$, $\gamma > 0$, $\sup_{t \geq t_0} (t - h(t)) < \infty$.

This equation with a constant concentrated delay and constant coefficients

$$
\frac{dN}{dt} = \frac{rN_{\tau}}{1 + N_{\gamma}^2} - bN,
$$

for the first time was developed in [26] to model white blood cells production. Here $N(t)$ is the density of mature cells in blood circulation, the function $\frac{rN_{\tau}}{1 + N_{\gamma}^2}$ models the blood cell reproduction, the time lag $N_{\tau} = N(t - \tau)$ described the maturational phase before blood cells are released into circulation, the mortality rate $bN$ was assumed to be proportional to the circulation. Equation (42) was introduced to explain the oscillations in numbers of neutrophils observed in some cases of chronic myelogenous leukemia [26, 27]. Here we assume that the maturation time is not constant but is distributed around some expectancy value.

For a modification of equation (42) to variable delays and coefficients, positivity of solutions and global asymptotic stability was recently studied in [28]; local stability conditions for its modification with the delay in the mortality term, as well as in the production term, were presented in [29]. Various aspects of equations of type (42) with variable parameters were discussed in [30, 31]. Stability of equations of population dynamics with integral type delays (both finite and infinite) was studied in [32].

The reproduction function can differ from one in (42): for instance, $\frac{r}{K^\gamma + N^\gamma}$ describes the red blood cells production rate [33], where three parameters $r, K, \gamma$ are chosen to match experimental data.

The linearized equation for (41) has the form

$$
\dot{y}(t) = A(t) \int_{h(t)}^t y(s) d_s R(t, s) - B(t)y(t).
$$

Thus Part 1 of Corollary 1 yields the following statement.

**Theorem 5.** If there exists $\beta > 0$, $\gamma \in (0, 1)$ such that $B(t) \geq \beta$, $A(t) \leq \gamma A(t)$ then the zero solution of (41) is locally uniformly asymptotically stable.

Consider now equation (41) with proportional coefficients

$$
\dot{N}(t) = r(t) \left[ \int_{h(t)}^t \frac{AN(s)}{1 + [N(s)]^\gamma} d_s R(t, s) - BN(t) \right], \quad t \geq 0,
$$

where $A > 0$, $B > 0$ and $r(t) \geq \alpha > 0$.

Applying Theorem 5 to equation (44) gives the following result.
COROLLARY 4. If \( B > A \) then the zero solution of (44) is locally uniformly asymptotically stable.

If \( A > B \) then equation (44) has the positive equilibrium

\[
N^* = \left( \frac{A}{B} - 1 \right)^{\frac{1}{\gamma}}.
\]

Let us now derive local stability conditions of the positive steady state \( N^* \).

To this end substitute \( N(t) = N^*(1 + x(t)) \) in (44); we obtain the following equation

\[
\dot{x}(t) = r(t) \left[ \int_{h(t)}^{t} \frac{A(1 + x(s))}{1 + [N^*(1 + x(s))]^\gamma} d_s R(t, s) - B(1 + x(t)) \right], \ t \geq 0.
\]

Since the derivative of the function

\[
z(u) = \frac{A(1 + u)}{1 + [N^*(1 + u)]^\gamma}
\]

satisfies

\[
z'(0) = -\frac{(A - B - A)B}{A},
\]

then by Corollaries 3, 2, Lemma 6 and Theorem 1 we obtain the following result.

THEOREM 6. Suppose that \( A > B > 0 \) and at least one of the following conditions holds:

1) \( \frac{|\gamma(A - B) - A|}{A} < 1 \);

2) \( \frac{|\gamma(A - B) - A|B}{A} \limsup_{t \to \infty} \int_{h(t)}^{t} \left( \int_{s}^{t} r(\tau) d\tau \right) d_s R(t, s) < \frac{\gamma(A - B)}{A + |\gamma(A - B) - A|} \);  

3) \( \gamma > \frac{A}{A - B} \frac{\gamma(A - B)B}{A} \limsup_{t \to \infty} \int_{h(t)}^{t} r(s) ds < U \approx 1.425 \).

4) \( \gamma > \frac{A}{A - B} \frac{(\gamma(A - B) - A)B}{A} \limsup_{t \to \infty} \int_{h(t)}^{t} r(s) ds < 1 + \frac{1}{e} \).

Then for equation (44) the positive steady state \( N^* \) defined by (45) is locally uniformly asymptotically stable.
REFERENCES


