ON POSITIVITY OF GREEN FUNCTIONS FOR A FUNCTIONAL-DIFFERENTIAL EQUATION

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Abstract. A necessary and sufficient condition of negativity of the Green function of the problem

\[ u'''(x) - \int_0^l u(s)d_x r(x, s) = f(x), \ x \in [0, l], \]

\[(u(0), u'(0), u(l)) = 0\]

in terms of spectral radii of two auxiliary problems.

Key Words. Positivity of Green function, zeros, non-oscillation

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1. The problem and main result.

1.1. The problem. We study the question about negativity of the Green function of the boundary value problem

\[ Lu(x) := u'''(x) - \int_0^l u(s)d_x r(x, s) = f(x), \ x \in [0, l], \]

\[ B(u) := (u(0), u'(0), u(l)) = 0, \]
(the symbol := means is equal by definition to) under the assumptions given in the subsection 1.2. The main result is the theorem 1 which establishes a necessary and sufficient conditions of such negativity in terms of the first eigenvalues of two auxiliary boundary value problems. These eigenvalues can be efficiently estimated to obtain efficient conditions of negativity.

For analogous delay equation the problem (1),(2) was investigated in [9]. On the other hand the presented work can be considered as continuation of the work [6]. For delay equation it will be also mentioned [4], [8].

1.2. Assumptions, remarks. Assume that $r(x,s)$ is nondecreasing in $s$ for almost all $x \in [0,l]$, $r(x,0) = 0$, $r(x,s)$ is measurable for all $s$ and $r(x,l)$ is Lebesgue integrable on $[0,l]$. The function $f(x)$ is assumed to be integrable on $[0,l]$. Solution $u(x)$ is a function with absolutely continuous second derivative $u''$ satisfying (1) for almost all $x \in [0,l]$.

Note that the deviating equation of the form

$$u''(x) - \sum_i p_i(x)u(h_i(x)) = f(x)$$

is a particular case of (1). Here the functions $p_i(x)$ are Lebesgue integrable on $[0,l]$, and the functions $h_i(x)$ are measurable (it is assumed, that if $h(x) \notin [0,l]$, $u(h(x)) = 0$). The nondecreasing condition for the deviating equation means the positiveness $p_i(x) \geq 0$. The case when $r(x,s)$ is non-increasing is more simple for the problem (1),(2). It was investigated, for example, in [5].

1.3. About representation of solutions of general boundary value problem. Let $L = L(0,l)$ be the space of Lebesgue integrable on $[0,l]$ functions with the norm $\|z\| = \int_0^l |z(s)| \, ds$. By $D$ denote the domain of the operator $L$ (defined by (1)), which is the set of all functions with absolutely continuous derivative $u''$ and the norm

$$\|u\| = \int_0^l |u''(s)| \, ds + |u(0)| + |u'(0)| + |u''(0)|.$$

For any linear (bounded) functional $B: D \to R^3$ the linear boundary value problem

$$Lu = f, \ B(u) = \alpha,$$

where $f \in L$, $\alpha \in R^3$, is Fredholm one [1]. If this problem is uniquely solvable, its solution has representation

$$u = Gf + U\alpha,$$
where the Green operator $G$ has integral representation $\int_0^l G(x,s)f(s)\,ds$, and
\[ U\alpha(x) = \alpha_0 u_0(x) + \alpha_1 u_1(x) + \alpha_2 u_2(x). \]
Thus, in expanded form (4) can be written as
\[ u(x) = \int_0^l G(x,s)f(s)\,ds + \alpha_0 u_0(x) + \alpha_1 u_1(x) + \alpha_2 u_2(x). \]

1.4. Positive solvability. The positivity\(^1\) of the Green’s function $G(x,s)$ for the problem (3) is equivalent to positivity of the Green’s operator $G$. Usually, but not always, the operator $U$ is also positive. This means that $u_0 \geq 0$, $u_1 \geq 0$, $u_2 \geq 0$. We mean positivity according to the following definition.

**Definition 1 (Positive solvability).** Say that the problem (3) is positively resolvable, if it is uniquely resolvable for any $f \in L(0,l)$ and $\alpha \in \mathbb{R}^3$ and
\[ (f \geq 0, \alpha \geq 0) \rightarrow u \geq 0. \]
It is clear that positive solvability of (3) is equivalent to non-negativity of the Green function $G(x,s)$ and non-negativity of the functions $u_0(x)$, $u_1(x)$, $u_2(x)$.

1.5. Main result. Consider the equation
\[ u'''(x) - \lambda \int_0^l u(s)r(x,s)\,ds = 0 \tag{5} \]
and the boundary conditions
\[ B_0(u) := (u(0), u'(0), u''(0)) = 0, \tag{6} \]
\[ B_1(u) := (u(0), u(l), -u'(l)) = 0. \tag{7} \]
Let $\lambda^{(0)}$ and $\lambda^{(l)}$ be the smallest positive eigenvalues of problems (5),(6) and (5),(7) respectively. If one of them does not exist, it is assumed to be equal to infinity.

The main result is presented in the following theorem.

**Theorem 1.** The condition
\[ (\lambda^{(0)} > 1) \land (\lambda^{(l)} > 1) \tag{8} \]
is necessary and sufficient for:

\(^1\) we use this term as nonnegativity
1. the problem

\( \mathcal{L}u = f, \ B(u) = (u(0), u'(0), u(l)) = \alpha \)

is uniquely solvable for any \( f \in L \), and \( \alpha \in \mathbb{R}^3 \).

2. let \( u \) be the solution of the problem (9); if \( f \geq 0, \ \alpha = (0, \alpha_1, \alpha_2), \ \alpha_1 \leq 0, \ \alpha_2 \leq 0 \), then \( u \leq 0 \), and if \( \int_0^l f(x) \, dx - \alpha_1 - \alpha_2 > 0 \), then \( u(x) < -cx^2(l - x) \) for some \( c > 0 \).

Remark 1. In the second assertion the \( \alpha = (0, \alpha_1, \alpha_2) \) cannot be arbitrary. For example, equation \( u''' = 1 \) has solution \( u = -(x - 2)^3/6 + 2(x - 2) \), satisfying \( u(0) < 0, u'(0) = 0, u(l) = 0 \) for \( l = \sqrt{12} \). This solution changes sign, in spite of \( \lambda^{(0)} = \lambda^{(l)} = \infty \).

2. Auxiliary problems. Here we study two auxiliary problems with conditions (6) and (7). Consider two problems

\( \mathcal{L}u = f, \ B_0(u) = \alpha = (\alpha_0, \alpha_1, \alpha_2) \)

and

\( \mathcal{L}u = f, \ B_l(u) = \beta = (\beta_0, \beta_1, \beta_2) \).

The scheme of investigation of these problems is well known (see for example [10], [5], [7], [1]). The basic idea consists in reducing of the boundary value problem to an integral equation and in estimation of the spectral radius of the integral operator. For this assessment are used the theorems on differential and integral inequalities.

2.1. Reducing to integral equations. Consider two problems

\( u''' = z, \ B_0(u) = \alpha \)

and

\( u''' = z, \ B_l(u) = \beta \).

They have the solutions

\( u = H_0z + V_0\alpha \)

and

\( u = H_lz + V_l\beta \)

respectively. Here

\( H_0z(x) = \int_0^x \frac{(x - s)^2}{2} z(s) \, ds, \)
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\[ V_0 \alpha(x) = \alpha_0 + \alpha_1 x + \alpha_2 \frac{x^2}{2}, \]

\[ H_l z(x) = \int_0^l H_l(x, s) z(s) \, ds \]

is the solution of \( u''' = z, \) \( B_0(u) = 0, \) and \( V_l \beta(x) \) is the solution of \( u''' = 0, \)
\( u(0) = \beta_0, \) \( u(l) = \beta_1, -u'(l) = \beta_2. \)

Denote

\[ Qu(x) = \int_0^l u(s) \, ds r(x, s). \]

Substituting (12) and (13) in (10) and (11) we will have the equations

(14) \[ z = QH_0 z + QV_0 \alpha + f \]

and

(15) \[ z = QH_l z + QV_l \beta + f \]

respectively. These equations are integral equations. In fact, for example,

\[ QH_l(x) = \int_0^l d_t r(x, t) \int_0^l H_l(t, s) z(s) \, ds = \int_0^l K_l(x, s) z(s) \, ds \]

where

\[ K_l(x, s) = \int_0^l H_l(t, s) d_t r(x, t). \]

The operator \( K_l \) with the kernel \( K_l(x, s) \) acts in \( L(0,l) \) and is compact [1].

The operators \( K_0 = QH_0 \) and \( K_l = QH_l \) are positive (and their kernels are non-negative). To verify this, the easiest way is to note that the solutions of the problems \( u''' = z \) with conditions \( B_0(u) = 0 \) and \( B_l(u) = 0 \) are nonnegative, if \( z \geq 0. \) After that we use the nondecreasing condition of \( r(x, s) \) in \( s. \)

We will use the following important properties of the operators \( H_0 \) and \( H_l, \) that can be verified simply.

**Lemma 1.** If \( z \geq 0, \) \( z \neq 0, \) and \( u = H_0 z, \) then \( u(x) \geq 0 \) and \( u'(x) \geq 0, \)
\( x \in [0,l]. \)

**Proof.** Function \( u = H_0 z \) is the solution of \( u''' = z, \) \( B_0(u) = 0. \) From this, \( u'' \geq 0, \) \( u' \geq 0 \) and \( u \geq 0. \)

**Lemma 2.** If \( z \geq 0, \) \( z \neq 0, \) then \( H_l z(x) \geq cx(l-x)^2 \) for some \( c > 0. \)

In (12) and (13) \( V_0 \alpha \) and \( V_l \beta \) are positive on \((0,l)\) if \( \alpha \) and \( \beta \) are nontrivial nonnegative. So the following propositions are valid.
Lemma 3. If spectral radius $\rho(QH_0) < 1$, then the problem (10) is positively resolvable. Moreover, if $f \geq 0$, and $\alpha \geq 0$, then $u(x) \geq 0$ and $u'(x) \geq 0$.

Proof. The solution of the problem (10) is $u = H_0z + V_0\alpha$, where

$$z = (I - K_0)^{-1}(f + QV_0\alpha).$$

So $z \geq 0$. Now see the lemma 1.

Lemma 4. If spectral radius $\rho(QH_I) < 1$, then the problem (11) is positively resolvable. Moreover, if $f \geq 0$, $\beta \geq 0$ and $\int_0^l f(s)ds + \beta_0 + \beta_1 + \beta_2 > 0$, then $u(x) \geq cx(l - x)^2$ for some $c > 0$.

To obtain effective conditions for positive solvability it can be used a theorem about estimation of spectral radius of positive operator. Such theorems are developed in different investigations. See for example, [2]. But in our situation it is more convenient to use theorems from [10], [5].

2.2. Theorems on differential inequalities.

Theorem 2 (Differential inequality 1). Suppose there exists a nonnegative solution of the inequalities \( Lu = \psi \geq 0 \), $B_0(u) \not\equiv 0$. Then $\rho(QH_0) < 1$.

Proof. Note that spectral radii of operators $QH_0$ and $H_0Q$ coincide. Since

$$u'' = Qu + \psi = z, \quad u = H_0z + V_0\alpha, \quad \alpha = B_0(u)$$

and

$$u - H_0Qu = H_0\psi + V_0\alpha.$$

Since $Lu = \psi \geq 0$, $\alpha = B_0(u) \not\equiv 0$, the right side is greater than or equal to $V_0\alpha \geq \gamma x^2 = \gamma u_0$.

Now, we can refer to the lemma 11, because the cone of all nonnegative functions is almost reproducing in $D$, and the operator $H_0Q$ is $u_0$-upper bounded where $u_0(x) = x^2$. 

Now consider the second problem (11).

Theorem 3 (Differential inequality 2). Suppose there exists a nonnegative solution of the inequalities \( Lu = \psi \geq 0 \), $B_1(u) \geq 0$, and $\int_0^l \psi(s)ds + u(0) + u(l) - u'(l) > 0$. Then $\rho(QH_I) < 1$.

Proof. As in theorem 2 we have

$$u - H_IQu = H_I\psi + V_I\beta,$$

where $\beta = B_1(u)$. The operator $H_IQ$ is $u_0$-positive, where $u_0(x) = x(l - x)^2$. Now we refer to the lemma 12.
2.3. Another auxiliary problem. We need to use one more auxiliary problem with a parameter $\xi \in (0, l]$. It is considered in the same spaces $L(0, l)$ and $D$.

\[
Lu = f, \quad B_\xi(u) = 0,
\]

where $B_\xi(u) = (u(0), u(\xi), u'(\xi))$. To justify such a designation, note that the problem (11) with $\beta = 0$ coincides with the (16), when $\xi = l$. Moreover, let $u = H_\xi z$ be the solution of the problem $u''' = z, B_\xi(u) = 0$. One can to verify that $\lim_{\xi \to 0} H_\xi z = H_0 z$, where the convergence is understood in the sense of the space $D$. So, the (10) with $\alpha = 0$ is the limit case of (16), when $\xi \to 0$. Thus, the problem (16) includes the previous (10) and (11) (except for nonzero boundary conditions).

The substitution $u = H_\xi z$ reduces the (16) to the equation

\[
z - QH_\xi z = f
\]

with integral operator $K_\xi = QH_\xi$ with nonnegative kernel $K_\xi(x, s)$. From here we have the following proposition.

**Lemma 5.** Suppose $\xi > 0$, and the spectral radius $\rho(K_\xi)$ of the operator $K_\xi$ is less then unity. If $f \geq 0$, then the problem (16) has unique nonnegative solution $u$, and $u(x) \geq cx(x - \xi)^2, x \in [0, l]$, for some $c > 0$.

It can be verified in the same manner as the lemma 4.

2.4. Truncated equation. Consider the truncated equation on the segment $[0, \xi]$

\[
L_\xi u \equiv u''' - \int_0^\xi u(s) d_s r(x, s) = f, \quad x \in [0, \xi],
\]

and boundary conditions

\[
u(0) = 0, \quad u(\xi) = 0, \quad u'(\xi) = 0.
\]

Note that this problem differs from the problem for the equation $Lu = f$ with the same boundary conditions (19). It differs from the first problem not only in the operator, but also in the segment $[0, \xi]$.

Let $Q_\xi$ be defined by $Q_\xi u(x) = \int_0^\xi u(s) d_s r(x, s)$, and $H_\xi$ be the Green operator of the problem

\[
u''' = z, \quad u(0) = 0, u(\xi) = 0, u'(\xi) = 0.
\]

Denote $\hat{K}_\xi = Q_\xi \hat{H}_\xi$. Let $\rho(\hat{K}_\xi)$ be the spectral radius of the operator $\hat{K}_\xi$. It is clear that the theorem 3 and lemma 4 can be reformulated for this equation on $[0, \xi]$:
Theorem 4. Suppose there exists a nonnegative solution $u$ of $\mathcal{L}_\xi u = \psi \geq 0$, $u(0) \geq 0$, $u(\xi) \geq 0$, $u'(\xi) \leq 0$, and $\int_0^\xi \psi(s) \, ds + u(0) + u(\xi) - u'(\xi) > 0$. Then $\rho(\tilde{K}_\xi) < 1$.

Lemma 6. If spectral radius $\rho(\tilde{K}_\xi) < 1$ then the problem (18), (19) is positively resolvable.

Lemma 7. If $0 < \xi_1 < \xi_2 \leq l$, then $\rho(\tilde{K}_{\xi_1}) < \rho(\tilde{K}_{\xi_2})$.

Proof. For simplicity of notation consider the case of $2 = l$, $1 = \tilde{l}$, and $(K_{\tilde{l}}) = 1$. Let $z$ be the eigenfunction of $K_l$, $z = K_{\tilde{l}}z = QH_lz$. It is nonnegative by the theorem 10. Then $u = H_lz$ is positive solution of $\mathcal{L}u = 0$, $u(0) = 0$, $u(l) = 0$, $u'(l) = 0$.

Note that $u''(l) > 0$, so $u'(\xi) < 0$ for $\xi$ sufficiently close to $l$. Then $\mathcal{L}_\xi u = \int_\xi^l u(s) \, ds r(x,s) \geq 0$, $u(\xi) > 0$, $u'(\xi) < 0$. By virtue of the theorem 4 $\rho(K_\xi) < 1$. It is clear that this is valid for all $\xi$, not only for $\xi$ close to $l$. □


3.1. Solvability, nonoscillation.

Lemma 8. Suppose one of the spectral radii is less than one: $\rho(K_0) < 1$ or $\rho(K_l) < 1$. Then the problem (1), (2) is uniquely solvable.

Proof. Because of the Fredholm property we can show that the homogeneous problem $\mathcal{L}u = 0$, $u(0) = u'(0) = u(l) = 0$ only has trivial solution.

First, suppose $\rho(K_0) < 1$. Suppose $u''(0) \geq 0$ for definiteness. By the lemma 3 $u(x) \geq 0$, $u'(x) \geq 0$ on $[0, l]$. Since $u(l) = 0$, $u(x) \equiv 0$.

In the same manner can be showed the second part of the theorem. □

Theorem 5 (Nonoscillation). If $\rho(K_0) < 1$ and $\rho(K_l) < 1$, then any solution of the homogeneous equation $\mathcal{L}u = 0$ satisfying $u(0) = 0$ can have at most one simple zero in the interval $(0, l]$.

Fig. 1. To the theorem 5

Proof. Let $u_1(x)$ be the solution to the boundary value problem

$$\mathcal{L}u = 0, \quad u(0) = 0, \quad u(l) = 0, \quad u'(l) = 1.$$
By lemma 4 it is negative in $(0, l)$, and $u'(0) < 0$. Up to a factor any solution satisfying $u(0) = 0$ can be represented in the form
\[ u(x) = u_1(x) + Cu_2(x), \]
where $C$ is some constant, and $u_2(x)$ is the solution to the boundary value problem
\begin{equation}
(22) \quad Lu = 0, \quad u(0) = 0, \quad u'(0) = 0, \quad u''(0) = 1.
\end{equation}
The solution $u_2(x)$ of the problem (22) is positive on $(0, l)$ by the lemma 3.

If $C \leq 0$, $u(x)$ does not have zeros in $(0, l)$ because of $u(x) \leq u_1(x) < 0$. Suppose $C > 0$. Then $u(l) = u_1(l) + Cu_2(l) = Cu_2(l) > 0$, $u(0) = 0$ and $u'(0) = u_1'(0) < 0$. So, $u(x)$ has zeros in $(0, l)$. Let $x_2$ be the maximal zero in $(0, l)$.

Suppose $u'(x_2) > 0$ and $x_1 < x_2$, where $x_1$ the nearest zero to $x_2$. Let $v(x) = u(x) + Du_2(x)$ where
\[ D = \max_{x \in [x_1, x_2]} \left( -\frac{u(x)}{u_2(x)} \right) = -\frac{u(\xi)}{u_2(\xi)}, \quad \xi \in (x_1, x_2). \]

From here $v(x) \geq 0$ for $x \in [x_1, x_2]$, and $v(\xi) = v'(\xi) = 0$. Moreover, $v(x) > 0$ on $[x_2, l]$. Note that $v'(0) = u'(0) + Du_2'(0) < 0$.

Since
\[ Lv = 0, \quad v(0) = 0, \quad v(\xi) = v'(\xi) = 0, \]
the function $v$ is a solution of the truncated boundary value problem
\[ L_\xi v = \int_\xi^l v(s) d_\lambda r(x, s) \geq 0 \]
with boundary conditions (19). By the lemma 7 the spectral radius $\rho(K_\xi) < \rho(K_l) < 1$. By the lemma 6 $v(x) \geq 0$. But this contradicts to $v'(0) < 0$. This contradiction shows that $u(x) < 0$ on $(0, \xi)$.

The case $u'(x_2) = 0$ is similar to the above-considered for $\xi = x_2$. \hfill \Box

We will need a proposition to compare the spectral radii $\rho(K_\xi)$ with $\rho(K_0)$ and $\rho(K_l)$. For this aid consider the problem with parameter $\lambda$
\begin{equation}
(23) \quad u''' - \lambda Q u = 0, \quad B_\xi(u) = 0.
\end{equation}
By means of the substitution \( u = H_\xi z \) the problem (23) can be reduced to the eigenvalue problem \( z - \lambda K_\xi z = 0 \), or

\[
K_\xi z = \frac{1}{\lambda} z.
\]

The \( K_\xi : L(0, l) \to L(0, l) \) is compact. According to the Krein-Rutman theorem 10 the smallest \( \lambda \) for which (23) has a nonzero solution is equal to \( 1/\rho(K_\xi) \). Denote this \( \lambda \) by \( \lambda^{(\xi)} \). So,

\[
(24) \quad \lambda^{(\xi)} = \frac{1}{\rho(K_\xi)}.
\]

If such \( \lambda \) does not exist, assume \( \lambda^{(\xi)} = +\infty \). The value of \( \lambda \) is called regular if (23) has only the trivial solution.

**Lemma 9.** \( \lambda^{(\xi)} \geq \min\{\lambda^{(0)}, \lambda^{(l)}\} \).

**Proof.** The theorem 5 remains valid for the equation \( u''' - \lambda Qu = 0 \) with the parameter \( \lambda \). Therefore, if \( \lambda \rho(K_0) < 1 \) and \( \lambda \rho(K_l) < 1 \), then any nontrivial solution to the equation \( u''' - \lambda Qu = 0 \), and satisfying \( u(0) = 0 \), can have at most one simple zero in \( (0, l] \). This means, that the problem (23) has only the trivial solution, and the \( \lambda \) is regular.

But the inequalities \( \lambda \rho(K_0) < 1 \) and \( \lambda \rho(K_l) < 1 \) can be represented in the form \( \lambda < \lambda^{(0)} \) and \( \lambda < \lambda^{(l)} \). Thus, any \( \lambda < \min\{\lambda^{(0)}, \lambda^{(l)}\} \) is regular, and \( \lambda^{(\xi)} \) cannot satisfy \( \lambda < \min\{\lambda^{(0)}, \lambda^{(l)}\} \).

### 3.2. Negativity of solutions.

**Lemma 10.** Suppose that \( \rho(K_0) < 1 \) and \( \rho(K_l) < 1 \). Let \( u \) be the solution of (1),(2). If \( f(x) \geq 0 \), then \( u''(0) < 0 \) and \( u'(l) > 0 \).

**Proof.** The (1),(2) has unique solution (lemma 8). It is a nonzero solution to (10) with \( \alpha = (0, 0, u''(0)) \). Suppose \( u''(0) \geq 0 \). By lemma 3 \( u(x) \geq 0, \ u'(x) \geq 0 \). But \( u(l) = 0 \). From here \( u(x) \equiv 0 \). This contradiction shows that \( u''(0) < 0 \).

Suppose that \( u'(l) \leq 0 \). The \( u(x) \) is the solution to (11) with \( \beta = (0, 0, -u'(l)) \). By lemma 4 \( u(x) \geq 0 \), that contradicts to \( u(0) = u'(0) = 0, \ u''(0) < 0 \).

**Theorem 6 (Negativity).** Suppose that \( \rho(K_0) < 1 \) and \( \rho(K_l) < 1 \). If \( f(x) \geq \neq 0 \), then the problem (1),(2) is uniquely solvable, its solution is negative, and \( u(x) < -cx^2(l-x) \) in \( (0, l) \) for some \( c > 0 \).

**Proof.** Let \( u(x) \) be the solution of the problem (1),(2). From the lemma 10 it follows \( u(x) < 0 \) in some neighborhoods of the points 0 and \( l \).
Suppose that \( u(x_0) \geq 0 \) in some point \( x_0 \in (0, l) \). Let \( x_0 \) be the maximum point: \( u(x_0) = \max\{u(x) : x \in [0, l]\} \). In this case we first show, that there exists a solution of the problem (16), i.e. a solution having double zero in a point \( \xi \).

If \( u(x_0) = 0 \), then \( x_0 \) is multiple zero (because our solution is negative, and the maximal value of this solution is equal to zero). In this case the \( u \) itself is the required solution. If \( u(x_0) > 0 \), we can construct a non-positive solution of \( Lu = f \) having a multiple zero.

Let \( v(x) = u(x) - Cu_2(x) \), where \( u_2(x) \) is the solution of the problem (22), and

\[
C = \max_{(0,l)} \frac{u(x)}{u_2(x)} = \frac{u(\xi)}{u_2(\xi)}, \quad \xi \in (0,l).
\]

This maximum exists because \( u(x) \) is negative in some neighborhood of \( x = 0 \). The function \( v(x) \) is non-positive, as

\[
v(x) = u(x) - Cu_1(x) = u(x) - \frac{u(\xi)}{u_1(\xi)} u_1(x) \leq u(x) - \frac{u(x)}{u_1(x)} u_1(x) = 0.
\]

Thus, \( v(\xi) = v'(\xi) = 0 \), and the function \( v(x) \) is the solution of the problem (16).

By lemma 9 \( \rho(K_\xi) < 1 \). Thus, \( v(x) \) is positive by lemma 5. This contradiction proves negativity of \( u \) on \( (0,l) \).

The inequality \( u(x) < -cx^2(l-x) \) for some \( c > 0 \) follows from lemma 10.

**Theorem 7.** Suppose \( \rho(K_0) < 1 \) and \( \rho(K_1) < 1 \). Then the solution \( u \) of the problem

\[ Lu = 0, \quad B(u) = (u(0), u'(0), u(l)) = (0, \alpha, \beta) \geq 0 \]

(\( \alpha + \beta > 0 \)) is positive on \( (0,l) \), and \( u(x) > cx^2(l-x) \) for some \( c > 0 \).

**Proof.** The \( u \) is nontrivial if one of the numbers \( \alpha \) or \( \beta \) is positive. It is sufficient to consider the two cases: \( (0, \alpha, \beta) = (0, 1, 0) \) and \( (0, \alpha, \beta) = (0, 0, 1) \).
In the first case
\[ u(x) = \frac{1}{u_1'(0)}u_1(x), \]
where \( u_1(x) \) is the solution of the problem \( L u = 0, u(0) = 0, u(l) = 0, u'(l) = -1 \). Now we refer to the lemma 4, and take in mind that \( u'(l) < 0 \).

In the second case
\[ u(x) = \frac{1}{u_2(l)}u_2(x), \]
where \( u_2(x) \) is the solution of the problem \( L u = 0, u(0) = 0, u'(0) = 0, u''(0) = 1 \). Now we refer to the lemma 3, and take in mind that \( u''(0) > 0 \).

The theorems 6 and 7 can be combined into one statement

**Theorem 8.** If \( \rho(K_0) < 1 \) and \( \rho(K_l) < 1 \), then

1. the problem

\[ (25) \quad L u = f, \quad B(u) = \alpha \]

is uniquely solvable for any \( f \in L \), and \( \alpha \in \mathbb{R}^3 \),

2. if \( f \geq 0, \alpha = (0, \alpha_1, \alpha_2), \alpha_1 \leq 0, \alpha_2 \leq 0 \), then \( u \leq 0 \), and if

\[ \int_0^l f(x) dx - \alpha_1 - \alpha_2 > 0, \]

then \( u(x) < -cx^2(l-x) \) for some \( c > 0 \).

### 3.3. Necessary conditions for negativity.

Here we show that the converse of the theorem 8 is also true. Thus the condition \( (\rho(K_0) < 1 \) and \( \rho(K_l) < 1 \) is necessary and sufficient for positive solvability of the problem (1),(2).

Note that here positive solvability we mean according to the following definition.

**Definition 2.** The problem (9) is positive solvable, if it is uniquely solvable for any \( f \in L, \alpha \in \mathbb{R}^3 \), and

\[ \{ f \geq 0, \alpha = (0, \alpha_1, \alpha_2) \leq 0 \} \rightarrow u \leq 0, \]

where \( u \) is the solution of (1),(2).

**Theorem 9.** Suppose problem (9) is positive solvable, and if

\[ \int_0^l f(x) dx - \alpha_1 - \alpha_2 > 0, \]

then \( u(x) < -cx^2(l-x) \) for some \( c > 0 \).

Then \( \rho(K_0) < 1 \) and \( \rho(K_l) < 1 \).

**Proof.** Let \( v \) be the solution of the problem \( L u = 0, u(0) = 0, u'(0) = 1, u(l) = 0 \). Then this solution is positive on \((0, l)\), and \( u'(l) < 0 \). By virtue of the theorem 3 the \( \rho(K_l) < 1 \) holds.

Now, let \( v \) be the solution of the problem \( L u = 0, u(0) = 0, u'(0) = 0, u(l) = 1 \). Then this solution is positive on \((0, l)\], and \( u''(0) > 0 \). By virtue of the theorem 2 the \( \rho(K_0) < 1 \) holds. \( \square \)
3.4. Efficient conditions for negativity. We use the theorems 2 and 3. Putting in the theorem 2 $u(x) = x^3 + \varepsilon x$, we have the following result.

**Corollary 1.** If for some $\varepsilon > 0$

$$\text{ess sup} \ r(x, l) \leq \frac{6}{l^3 + \varepsilon l},$$

then $\rho(K_0) < 1$.

**Proof.** We have

$$6 - \int_0^l (s^3 + \varepsilon s)d_x r(x, s) \geq 6 - (l^3 + \varepsilon l)r(x, l) \geq 0.$$

To receive an estimate for $\rho(K_1)$ we use the theorem 3.

**Corollary 2.** If

$$\text{ess sup} \ r(x, l) \leq \frac{81}{2l},$$

then $\rho(K_1) < 1$ (except for the case when $\int_0^l u(s)d_x r(x, s) \equiv \frac{81}{2l} u(l/3)$).

**Proof.** Let $u(x) = x(l - x)^2$. Then

$$Lu = 6 - \int_0^l s(l - s)^2d_x r(x, s) \geq 6 - \frac{4l}{2l} r(x, l) \geq 0.$$

**A. Auxiliary propositions.** Let $K$ be an almost reproducing cone\(^2\) in a Banach space $E$ and $A: E \to E$ be a linear compact operator. Assume that it is positive with relation to $K$, i.e. $AK \subset K$. Let $\rho(A)$ (or simply $\rho$) be spectral radius of the operator $A$. About notions in this section see for example [2].

**Theorem 10 (Krein, Rutman [3]).** If spectrum of $A$ contains different from zero points, then its spectral radius $\rho$ is an eigenvalue of $A$, this eigenvalue is simple, and if $Av_0 = \rho v_0$, then $v_0 \in K$.

**Definition 3.** Operator $A: E \to E$ is called $u_0$-upper bounded, if for any $x \in E$ there exists $\beta > 0$ such that $Ax \leq \beta u_0$.

Operator $A$ is called $u_0$-positive, if for any $x \in K$ there exist $\alpha > 0$, $\beta > 0$ such that $\alpha u_0 \leq Ax \leq \beta u_0$.

**Lemma 11 ([10]).** Suppose $A$ is $u_0$-upper bounded, where $u_0 \in K$, and there exists $v \in K$ satisfying the inequality $v - Av \geq \gamma u_0$ for a $\gamma > 0$. Then $\rho < 1$.

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\(^2\) $K$ is almost reproducing cone if the closure of its span is the whole space $E$
Lemma 12 ([10]). Suppose $A$ is $u_0$-positive, where $u_0 \in K$, and there exists $v \in K$ satisfying the inequality $v - Av = g \geq 0$. Then $\rho < 1$.

Proof. From $v - Av = g$ we have $v - A^2v = Ag + g \geq \alpha u_0$ for an $\alpha > 0$. From lemma 11 we have $\rho(A^2) < 1$. So $\rho < 1$.

REFERENCES