Oscillation Theorems for Second-Order Nonlinear Neutral Differential Equations of Mixed Type

E. Thandapani * and R. Rema †

Abstract. Some oscillation criteria are presented for the second-order nonlinear neutral differential equations of mixed type

\[ [x(t) + p_1 x(t - \tau_1) + p_2 x(t + \tau_2)]'' = q_1(t)x^{\beta}(t - \sigma_1) + q_2(t)x^{\gamma}(t + \sigma_2), \ t \geq t_0 \]

where \( \alpha, \beta \) and \( \gamma \) are the ratio of odd positive integers. These criteria generalize and complement to existing results. Examples are provided to illustrate the main results.

Key Words. Second-Order, Oscillation, Neutral Mixed Type

AMS(MOS) subject classification. 34C15

1. Introduction. This paper concerned with the oscillatory behavior of second-order nonlinear neutral differential equation of mixed type

\[ [(x(t) + p_1 x(t - \tau_1) + p_2 x(t + \tau_2))'' = \ q_1(t)x^{\beta}(t - \sigma_1) + q_2(t)x^{\gamma}(t + \sigma_2), \ t \geq t_0. \]

(1.1)

subject to the following conditions:

\( (H_1) \) \( p_i, \tau_i \) and \( \sigma_i, \ i = 1, 2 \) are positive constants;

---

* Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai -600 005
† Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai -600 005.
(H_2) q_i \in C([t_0, \infty), [0, \infty)), \ i = 1, 2.

By a solution of equation (1.1), we mean a function \( x \in C([T_x, \infty), \mathbb{R}) \) for some \( T_x \geq t_0 \) which has the property that 
\[
(x(t) + p_1 x(t - \tau_1) + p_2 x(t + \tau_2))^\alpha \in C^2([T_0, \infty), \mathbb{R})
\]
and satisfies the equation (1.1) on \([T_x, \infty)\). As is customary, a solution of equation (1.1) is called oscillatory if it has arbitrarily large zeros on \([t_0, \infty)\), otherwise it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Neutral functional differential equations have numerous applications in electric networks. For instance, they are frequently used for the study of distribution networks containing lossless transmission lines which rise in high speed computers where the lossless transmission lines are used to interconnect switching circuits; see [13, 15].

In recent years, many results have been obtained on oscillation of different classes of functional differential equations, we refer the reader to the papers [1–12, 14, 16–20] and the references cited therein. Philos [17] established some Philos-type oscillation criteria for the second-order linear differential equation
\[
(r(t)x(t))' + q(t)x(t) = 0, \ t \geq t_0.
\] (1.2)
In [7], the authors gave some sufficient conditions for oscillation of all solutions of second-order half-linear differential equation
\[
(r(t)|x|^\gamma x'(t))'^\gamma x(\tau(t)) = 0, \ t \geq t_0
\] (1.3)
by employing a Riccati substitution technique.

Some oscillation criteria for the following second-order quasilinear neutral differential equation
\[
(r(t)|z|^\gamma z'(t))'^\gamma x(\sigma(t)) = 0,
\] (1.4)
for \( z(t) = x(t) + p(t)x(\tau(t)), \ t \geq t_0 \) were obtained by [9, 15].

However, there are few results regarding the oscillatory properties of neutral differential equations with mixed arguments. In [5, 14, 19], the authors established some oscillation criteria for the following mixed neutral equation

\[
(x(t) + p_1 x(t - \tau_1) + p_2 x(t + \tau_2))'' = q_1(t)x(t - \sigma_1) + q_2(t)x(t + \sigma_2), \ t \geq t_0
\]

(1.5)

with \( q_1 \) and \( q_2 \) are nonnegative real valued functions. Grace [11] obtained some oscillation theorems for the odd order neutral differential equation

\[
(x(t) + p_1 x(t - \tau_1) + p_2 x(t + \tau_2))^{(n)} = q_1 x(t - \sigma_1) + q_2 x(t + \sigma_2), \ t \geq t_0
\]

(1.6)

where \( n \geq 1 \) is odd. Grace [12] and Yan [21] obtained several sufficient conditions for the oscillation of solutions of higher-order neutral functional differential equation of the form

\[
(x(t) + cx(t - h) + Cx(t + H))^{(n)} + qx(t - g) + Qx(t + G) = 0, \ t \geq t_0
\]

(1.7)

where \( q \) and \( Q \) are nonnegative real constants.

Clearly, equations (1.5) and (1.6) when \( n=2 \), are special case of equation (1.1). Motivated by the above observation in this paper we study the oscillatory behavior of equation (1.1). The results obtained here generalize and complement to the results given in [5, 11, 12, 14, 19, 21]. Examples are provided to illustrate the main results.

In the sequel, when we write a functional inequality without specifying its domain of validity and we assume that it holds for all sufficiently large \( t \).

2. Main Results. In this section we establish oscillation criteria for the equation (1.1). We begin with two lemmas which will be used to prove
the main results.

**Lemma 1.** Assume that $0 < \gamma \leq 1$, and $x_1, x_2 \in [0, \infty)$. then

$$x_1^\gamma + x_2^\gamma \geq (x_1 + x_2)^\gamma. \quad (2.1)$$

**Lemma 2.** Assume that $\gamma \geq 1$, and $x_1, x_2 \in [0, \infty)$. Then

$$(x_1 + x_2)^\gamma \leq 2^{\gamma-1}(x_1^\gamma + x_2^\gamma). \quad (2.2)$$

The proofs of Lemmas 1 and 2 may be found in [14, 19].

**Lemma 3.** Let $\delta > 1$ be a ratio of odd positive integers and $\sigma \geq 2$ be a positive real number. If

$$\int_{t_0}^t \int_{s = t - \sigma + 1}^t R(s) ds dt = \infty, \quad (2.3)$$

then the differential inequality

$$y''(t + \sigma)$$

where $R(t)$ is a continuous function, has no eventually positive increasing solution.

**Proof.** Let $y(t)$ be an eventually positive increasing solution of

$$y''(t) \geq R(t)y^\delta(t + \sigma).$$

Then $y'(t) > 0$ for all $t \geq t_1 \geq t_0$. Integrating the last inequality from $t - \sigma + 1$ to $t$ we obtain

$$y'(t) \geq y'(t) - y'(t - \sigma + 1) \geq \int_{s = t - \sigma + 1}^t R(s)y^\delta(s + \sigma) ds$$

$$\geq y^\delta(t + 1) \int_{s = t - \sigma + 1}^t R(s) ds$$
or

\[ \frac{y'(t)}{y^\delta(t+1)} \geq \int_{s=t-\sigma+1}^{t} R(s) ds \]

or

\[ \frac{y'(t)}{y^\delta(t)} \geq \frac{y'(t)}{y^\delta(t+1)} \geq \int_{s=t-\sigma+1}^{t} R(s) ds. \]

Again integrating the last inequality from \( t_0 \) to \( t \), we get

\[ \frac{y^{1-\delta}(t_0)}{1-\delta} - \frac{y^{1-\delta}(t)}{1-\delta} \geq \int_{u=t_0}^{t} \int_{s=\sigma+1}^{u} R(s) ds du. \]

Letting \( t \to \infty \), we see that

\[ \int_{t=t_0}^{t} \int_{s=t-\sigma+1}^{t} R(s) ds dt < \infty \]

which contradicts (2.3). This completes the proof of the Lemma.

\[ \blacksquare \]

**Lemma 4.** Let \( \delta < 1 \) and \( \sigma \geq 1 \) be a positive real number. If

\[ \int_{t=t_0}^{\infty} \int_{s=t}^{t+\sigma} R(s) ds dt = \infty, \quad (2.4) \]

then the differential inequality

\[ y^{\sigma\delta}(t - \sigma) \]

has no eventually positive decreasing solution.

**Proof.** Let \( y(t) \) be an eventually positive decreasing solution of

\[ y''(t) \geq R(t)y^\delta(t - \sigma). \]

Then \( y'(t) < 0 \) for all \( t \geq t_1 \geq t_0 \).
Integrating the last inequality from \( t \) to \( t + \sigma \), we get

\[
-y'(t) \geq y'(t + \sigma) - y'(t) \geq \int_{s=t}^{t+\sigma} R(s)y^\delta(s - \sigma) \, ds
\]

or

\[
-\frac{y'(t)}{y^\delta(t)} \geq \int_{s=t}^{t+\sigma} R(s) \, ds.
\]

Again integrating from \( t_0 \) to \( t \), we get

\[
\frac{y^{1-\delta}(t_0)}{1 - \delta} \geq \frac{y^{1-\delta}(t_0) - y^{1-\delta}(t)}{1 - \delta} \geq \int_{u=t_0}^{t+\sigma} \int_{s=u}^{t} R(s) \, ds \, du.
\]

Letting \( t \to \infty \), we see that

\[
\int_{t=t_0}^{\infty} \int_{s=t}^{t+\sigma} R(s) \, ds \, dt < \infty
\]

which is a contradiction to (2.4). This completes the proof. \( \blacksquare \)

Now, we present oscillation criteria for the equation (1.1). For simplicity we introduce the following notations throughout this paper without further mention. \( Q_i(t) = \min\{q_i(t - \tau_1), q_i(t), q_i(t + \tau_2)\} \) for \( i = 1, 2 \) and \( p = 1 + p_1^\beta + p_2^\beta \) and \( q = 1 + p_1^\beta + \frac{1}{2^{\beta-1}}p_2^\beta \).

**Theorem 2.1.** Assume that \( 0 < \beta = \gamma \leq 1 \), and \( \sigma_i > \tau_i \) for \( i = 1, 2 \). If the differential inequality

\[
 y''(t) \geq \frac{Q_2(t)}{p^{\beta/\alpha}} y^{\beta/\alpha}(t - \tau_2 + \sigma_2)
\]

has no positive increasing solution, and the differential inequality

\[
 y''(t) \geq \frac{Q_1(t)}{p^{\beta/\alpha}} y^{\beta/\alpha}(t + \tau_1 - \sigma_1)
\]

(2.5b)
has no positive decreasing solution, then every solution of equation (1.1) is oscillatory.

**Proof.** Let \( x(t) \) be a nonoscillatory solution of equation (1.1). Without loss of generality, we assume that there exists \( t_1 \geq t_0 \) such that \( x(t) > 0, x(t - \tau_1) > 0 \) and \( x(t - \sigma_1) > 0 \) for all \( t \geq t_1 \). Setting

\[
z(t) = (x(t) + p_1 x(t - \tau_1) + p_2 x(t + \tau_2))^\alpha,
\]

and

\[
y(t) = z(t) + p_1^\beta z(t - \tau_1) + p_2^\beta z(t + \tau_2)
\]

we have \( z(t) > 0, y(t) > 0 \) and

\[
z''(t) = q_1(t)x^\beta(t - \sigma_1) + q_2(t)x^\beta(t + \sigma_2) \geq 0.
\]

Then \( z'(t) \) is of one sign, eventually. On the other hand,

\[
y''(t) = z''(t) + p_1^\beta z''(t - \tau_1) + p_2^\beta z''(t + \tau_2) = q_1(t)x^\beta(t - \sigma_1) + q_2(t)x^\beta(t + \sigma_2) + p_1^\beta [q_1(t - \tau_1)x^\beta(t - \tau_1 - \sigma_1) + q_2(t - \tau_1)x^\beta(t - \tau_1 + \sigma_2)] + p_2^\beta [q_1(t + \tau_2)x^\beta(t + \tau_2 - \sigma_1) + q_2(t + \tau_2)x^\beta(t + \tau_2 + \sigma_2)].
\]

(2.6)

Using the inequality (2.1), we have

\[
x^\beta(t - \sigma_1) + p_1^\beta x^\beta(t - \tau_1 - \sigma_1) \geq (x(t - \sigma_1) + p_1 x(t - \tau_1 - \sigma_1))^\beta,
\]

and

\[
(x(t - \sigma_1) + p_1 x(t - \tau_1 - \sigma_1))^\beta + p_2^\beta x^\beta(t + \tau_2 - \sigma_1) \geq z'^\alpha(t - \sigma_1).
\]
Therefore,

\[ x^\beta (t - \sigma_1) + p_1^\beta x^\beta (t - \tau_1 - \sigma_1) + p_2^\beta x^\beta (t + \tau_2 - \sigma_1) \geq z^{\beta / \alpha} (t - \sigma_1). \]

Similarly, we obtain

\[ x^\beta (t + \sigma_2) + p_1^\beta x^\beta (t - \tau_1 + \sigma_2) + p_2^\beta x^\beta (t + \tau_2 + \sigma_2) \geq z^{\beta / \alpha} (t + \sigma_2). \]

Thus from (2.6), we have

\[ y''(t) \geq Q_1 (t) z^{\beta / \alpha} (t - \sigma_1) + Q_2 (t) z^{\beta / \alpha} (t + \sigma_2). \] (2.7)

In the following we consider two cases:

**Case 1:** Assume that \( z'(t) > 0 \). Then \( y'(t) > 0 \). In view of (2.7), we see that

\[ y''(t + \tau_2) \geq Q_2 (t + \tau_2) z^{\beta / \alpha} (t + \tau_2 + \sigma_2). \]

Applying the monotonicity of \( z \), we get

\[ y(t + \sigma_2) = z(t + \sigma_2) + p_1^\beta z(t - \tau_1 + \sigma_2) + p_2^\beta z(t + \tau_2 + \sigma_2) \leq pz(t + \tau_2 + \sigma_2). \]

Combining the last two inequalities, we obtain

\[ y''(t + \tau_2) \geq \frac{Q_2 (t + \tau_2)}{p^{\beta / \alpha}} y^{\beta / \alpha} (t + \sigma_2). \]

Therefore, \( y \) is a positive increasing solution of the differential inequality

\[ y''(t) \geq \frac{Q_2 (t) y^{\beta / \alpha} (t - \tau_2 + \sigma_2)}{p^{\beta / \alpha}}, \] (2.8)

a contradiction to (2.5a).

**Case 2:** Assume that \( z'(t) < 0 \). Then \( y'(t) < 0 \). In view of (2.7), we see that

\[ y''(t - \tau_1) \geq Q_1 (t - \tau_1) z^{\beta / \alpha} (t - \tau_1 - \sigma_1). \]
Applying the monotonicity of \( z \), we get
\[
y(t - \sigma_1) = z(t - \sigma_1) + p_1^\beta z(t - \tau_1 - \sigma_1) + p_2^\beta z(t + \tau_2 - \sigma_1) \\
\leq p z(t - \tau_1 - \sigma_1).
\]
Combining the last two inequalities, we obtain
\[
y''(t - \tau_1) \geq \frac{Q_1(t - \tau_1)}{p^\alpha/\alpha} y^{\beta/\alpha}(t - \sigma_1).
\]
Therefore, \( y \) is a positive decreasing solution of the differential inequality
\[
y''(t) \geq \frac{Q_1(t)}{p^\alpha/\alpha} y^{\beta/\alpha}(t + \tau_1 - \sigma_1) \quad (2.9)
\]
which contradicts (2.5b).

This completes the proof of the theorem. \( \blacksquare \)

**Theorem 2.2.** Let \( \beta_i = \frac{\sigma_i - \tau_i}{2} > 0 \), \( i = 1, 2 \) and \( 0 < \beta = \gamma \leq 1 \). Suppose that for \( i = 1, 2 \) there exist functions
\[
a_i \in C^1[t_0, \infty), \ a_i(t) > 0, \ (-1)^i a'_i(t) \leq 0 \quad (2.10)
\]
such that
\[
Q_i(t) \geq p a_i(t) a_i(t + (-1)^i \beta_i). \quad (2.11)
\]
If the first order differential inequality
\[
V^{n+1} a_i(t + (-1)^i \beta_i) V(t + (-1)^i \beta_i) \geq 0 \quad (2.12)
\]
has no eventually negative solution for \( i = 1 \), and has no eventually positive solution for \( i = 2 \), then every solution of equation (1.1) is oscillatory.

**Proof.** Let \( x(t) \) be a nonoscillatory solution of equation (1.1). Without loss of generality, we assume that there exists \( t_1 \geq t_0 \) such that \( x(t) > 0, x(t - \tau_1) > \)
0, and \( x(t - \sigma_1) > 0 \) for all \( t \geq t_1 \). Define \( z(t) \) and \( y(t) \) as in Theorem 2.1. Proceeding as in the proof of Theorem 2.1, we get (2.7). Next, we consider two cases:

**Case 1:** Assume that \( z'(t) > 0 \). Then clearly \( y'(t) > 0 \). Then as in Case 1 of Theorem 2.1, we find that \( y \) is a positive increasing solution of inequality (2.8).

Let \( b_2(t) = y'(t) + a_2(t)y(t + \beta_2) \). Since \( y > 0 \) and \( y' > 0 \) we have \( b_2(t) > 0 \). Using (2.10) and (2.11), we get

\[
b_2'(t) - \frac{a_2'(t)}{a_2(t)} b_2(t) - a_2(t)b_2(t + \beta_2) = y''(t) + a_2'(t)y(t + \beta_2) + a_2(t)y'(t + \beta_2) - \frac{a_2'(t)}{a_2(t)} y'(t) - a_2'(t)y(t + \beta_2) - a_2(t)y'(t + \beta_2) - a_2(t)a_2(t + \beta_2)y(t + 2\beta_2) \geq y''(t) - \frac{Q_2(t)}{p}y(t - \tau_2 + \sigma_2) \geq 0.
\]

Define \( b_2(t) = a_2(t)V(t) \). Then \( V(t) > 0 \). Now

\[
V'(t) - a_2(t + \beta_2)V(t + \beta_2) = \frac{b_2'(t)}{a_2(t)} - \frac{a_2'(t)}{a_2(t)} b_2(t) - b_2(t + \beta_2) \geq 0.
\]

Therefore \( V(t) \) is a positive solution of the inequality (2.12) for \( i = 2 \), which is a contradiction.

**Case 2:** Assume that \( z'(t) < 0 \). Then clearly \( y'(t) < 0 \). As in the Case 2 of Theorem 2.1, we find that \( y \) is a positive decreasing solution of inequality (2.9).

Let \( b_1(t) = y'(t) - a_1(t)y(t - \beta_1) \). Then \( b_1(t) < 0 \).
Using (2.10) and (2.11), we have

\[
\begin{align*}
b'(t) - \frac{a'_1(t)}{a_1(t)} b_1(t) + a_1(t) b_1(t - \beta_1) &= y''(t) - \frac{a'_1(t)}{a_1(t)} y(t) \\
&\quad - a_1(t) a_1(t - \beta_1) y(t - 2\beta_1) \\
&\geq y''(t) - a_1(t) a_1(t - \beta_1) y(t - 2\beta_1) \\
&\geq y''(t) - \frac{Q_1(t)}{p} y(t - \tau_1 + \sigma_1) \geq 0.
\end{align*}
\]

Now define \(b_1(t) = a_1(t) V(t)\). Then \(V(t) < 0\).

Now

\[
V'(t) + a_1(t - \beta_1) V(t - \beta_1) = \frac{b'(t)}{a_1(t)} - \frac{a'_1(t)}{a^2_1(t)} + b_1(t - \beta_1) \geq 0.
\]

Therefore \(V(t)\) is a negative solution of the inequality (2.12) for \(i = 1\), which is a contradiction and the proof of the theorem is complete.

**Theorem 2.3.** Assume that \(\beta = \gamma \geq 1\), and \(\sigma_i > \tau_i\) for \(i = 1, 2\). If the differential inequality

\[
y''(t) \geq \frac{Q_2(t)}{(4^{\beta - 1}) q^{\beta / \alpha}} y^{\beta / \alpha}(t - \tau_2 + \sigma_2)
\]

has no positive increasing solution, and the differential inequality

\[
y''(t) \geq \frac{Q_1(t)}{(4^{\beta - 1}) q^{\beta / \alpha}} y^{\beta / \alpha}(t + \tau_1 - \sigma_1)
\]

has no positive decreasing solution, then every solution of equation (1.1) is oscillatory.

**Proof.** Let \(x(t)\) be a nonoscillatory solution of equation (1.1). Without loss of generality, we assume that there exists \(t_1 > t_0\) such that \(x(t) > 0, x(t - \tau_1) > 0,\) and \(x(t - \sigma_1) > 0\) for all \(t \geq t_1\). Setting

\[
z(t) = [x(t) + p_1 x(t - \tau_1) + p_2 x(t + \tau_2)]^\alpha,
\]
and
\[ y(t) = z(t) + p_1^\alpha z(t - \tau_1) + \frac{p_2^\beta}{2^{\beta-1}} z(t + \tau_2), \] \hspace{1cm} (2.14)

It is clear that \( z(t) > 0 \), \( y(t) > 0 \), and
\[ z''(t) = q_1(t)x^\beta(t - \sigma_1) + q_2(t)x^\beta(t + \sigma_2) \geq 0. \] \hspace{1cm} (2.15)

Therefore \( z'(t) \) is of one sign, eventually. On the otherhand
\[
\begin{align*}
y''(t) &= q_1(t)x^\beta(t - \sigma_1) + q_2(t)x^\beta(t + \sigma_2) + p_1^\alpha q_1(t - \tau_1)x^\beta(t - \tau_1 - \sigma_1) \\
&\quad + \frac{p_2^\beta}{2^{\beta-1}} q_1(t + \tau_2)x^\beta(t + \tau_2 - \sigma_1) \\
&\quad + \frac{p_2^\beta}{2^{\beta-1}} q_2(t + \tau_2)x^\beta(t + \tau_2 + \sigma_2).
\end{align*} \] \hspace{1cm} (2.16)

Using the inequality (2.2), we have
\[
x^\beta(t - \sigma_1) + p_1^\alpha x^\beta(t - \tau_1 - \sigma_1) \geq \frac{1}{2^{\beta-1}}(x(t - \sigma_1) + p_1x(t - \tau_1 - \sigma_1))^\beta,
\]
and
\[
\begin{align*}
\frac{1}{2^{\beta-1}}(x(t - \sigma_1) + p_1x(t - \tau_1 - \sigma_1))^\beta + \frac{1}{2^{\beta-1}} p_2^\beta x^\beta(t + \tau_2 - \sigma_1) \\
\geq \frac{z^{\beta/\alpha}(t - \sigma_1)}{(4^{\beta-1})}. \hspace{1cm} (2.17)
\end{align*}
\]

Similarly we obtain
\[
x^\beta(t + \sigma_2) + p_1^\alpha x^\beta(t - \tau_1 + \sigma_2) + \frac{p_2^\beta}{2^{\beta-1}} x^\beta(t + \tau_2 + \sigma_2) \geq \frac{z^{\beta/\alpha}(t + \sigma_2)}{(4^{\beta-1})}. \hspace{1cm} (2.18)
\]

Thus from (2.16), (2.17) and (2.18), we have
\[
y''(t) \geq \frac{1}{(4^{\beta-1})} \left( Q_1(t)z^{\beta/\alpha}(t - \sigma_1) + Q_2(t)z^{\beta/\alpha}(t + \sigma_2) \right). \hspace{1cm} (2.19)
\]

In the following, we consider two cases:
Case 1: Assume that \( z'(t) > 0 \). Then \( y'(t) > 0 \). In view of (2.19), we see that
\[
y''(t + \tau_2) \geq \frac{1}{(4^{\beta-1})} Q_2(t + \tau_2) z^{\beta/\alpha}(t + \tau_2 + \sigma_2). \tag{2.20}
\]
Applying the monotonicity of \( z \), we find that
\[
y(t + \sigma_2) = z(t + \sigma_2) + p_1 \beta z(t - \tau_1 + \sigma_2) + \frac{p_2}{2^{\beta-1}} z(t + \tau_2 + \sigma_2)
\leq qz(t + \tau_2 + \sigma_2). \tag{2.21}
\]
Combining the last two inequalities, we obtain
\[
y''(t + \tau_2) \geq \frac{Q_2(t + \tau_2)}{(4^{\beta-1})q^{\beta/\alpha}} y^{\beta/\alpha}(t + \sigma_2). \tag{2.22}
\]
Therefore \( y \) is a positive increasing solution of the differential inequality
\[
y''(t) \geq \frac{Q_2(t)}{(4^{\beta-1})q^{\beta/\alpha}} y^{\beta/\alpha}(t - \tau_2 + \sigma_2). \tag{2.23}
\]
which contradicts (2.13a).

Case 2: Assume that \( z'(t) < 0 \). Then \( y'(t) < 0 \). In view of (2.19) we see that
\[
y''(t - \tau_1) \geq \frac{1}{(4^{\beta-1})} Q_1(t - \tau_1) z^{\beta/\alpha}(t - \tau_1 - \sigma_1). \tag{2.24}
\]
Applying the monotonicity of \( z \), we find that
\[
y(t - \sigma_1) = z(t - \sigma_1) + p_1 \beta z(t - \tau_1 - \sigma_1) + \frac{p_2}{2^{\beta-1}} z(t + \tau_2 - \sigma_1)
\leq qz(t - \tau_1 - \sigma_1). \tag{2.25}
\]
Combining the last two inequalities, we obtain
\[
y''(t - \tau_1) \geq \frac{Q_1(t - \tau_1)}{(4^{\beta-1})q^{\beta/\alpha}} y^{\beta/\alpha}(t - \sigma_1). \tag{2.26}
\]
Therefore, \( y(t) \) is a positive decreasing solution of the differential inequality
\[
y''(t) \geq \frac{Q_1(t)}{(4^{\beta-1})q^{\beta/\alpha}} y(t + \tau_1 - \sigma_1). \tag{2.27}
\]
which contradicts (2.13b). This completes the proof. \( \blacksquare \)
Theorem 2.4. Let $\beta = \gamma \geq 1$, and $\beta_i = \frac{\sigma_i - \tau_i}{2} > 0$, $i = 1, 2$. Suppose that for $i = 1, 2$, there exist functions

$$a_i \in C^1([t_0, \infty)), \quad a_i(t) > 0, \quad (-1)^i a_i'(t) \leq 0$$

such that

$$Q_i(t) \geq (4^{\beta_i-1})q a_i(t) a_i(t + (-1)^i \beta_i).$$

If the first-order differential inequality

$$V^{n+1} a_i(t + (-1)^i \beta_i) V(t + (-1)^i \beta_i) \geq 0$$

has no eventually negative solution for $i = 1$ and no eventually positive solution for $i = 2$, then every solution of equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(t - \tau_1) > 0$, and $x(t - \sigma_1) > 0$ and for all $t \geq t_1$. Define $z$ and $y$ as in Theorem 2.3. Proceeding as in the proof of Theorem 2.3 we get (2.19).

In the following we consider two cases.

**Case 1:** Assume that $z'(t) > 0$. Then clearly $y'(t) > 0$. Then, as in Case 1 of Theorem 2.3, we find that $y$ is a positive increasing solution of inequality (2.23).

Let $b_2(t) = y'(t) + a_2(t) y(t + \beta_2)$. Then $b_2(t) > 0$. Using (2.28) and (2.29), we obtain

$$b_2'(t) - \frac{a_2'(t)}{a_2(t)} b_2(t) - a_2(t) b_2(t + \beta_2) = y''(t) - \frac{a_2'(t)}{a_2(t)} y'(t)$$

$$a_2(t) a_2(t + \beta_2) y(t + 2 \beta_2)$$

$$\geq y''(t) - a_2(t) a_2(t + \beta_2) y(t + 2 \beta_2)$$

$$\geq y''(t) - \frac{Q_2(t)}{(4^{\beta_2-1})q} y(t - \tau_2 + \sigma_2) \geq 0.$$  

(2.31)
Define \( b_2(t) = a_2(t)V(t) \). Then \( V(t) \) is a positive solution of (2.30) for \( i = 2 \), which is a contradiction.

**Case 2:** Assume that \( z'(t) < 0 \). Then clearly \( y'(t) < 0 \). Then, as in Case 2 of Theorem 2.3, we find that \( y \) is a positive decreasing solution of inequality (2.27). Let \( b_1(t) = y'(t) - a_1(t)y(t - \beta_1) \). Then \( b_1(t) < 0 \). Using (2.28) and (2.29), we obtain

\[
\begin{align*}
  b_1'(t) - \frac{a_1'(t)}{a_1(t)}b_1(t) + a_1(t)b_1(t - \beta_1) &= y''(t) - \frac{a_1'(t)}{a_1(t)}y'(t) \\
  &\quad - a_1(t)\frac{a_1(t - \beta_1)y(t - 2\beta_1)}{a_1(t)} \\
  &\geq y''(t) - a_1(t)a_1(t - \beta_1)y(t - 2\beta_1) \\
  &\geq y''(t) - \frac{Q_1(t)}{(4\beta_1 - 1)q}y(t + \tau_1 - \sigma_1) \geq 0.
\end{align*}
\]

(2.32)

Define \( b_1(t) = a_1(t)V(t) \). Then \( V(t) \) is a negative solution of (2.30) for \( i = 1 \). This contradiction completes the proof of the theorem. \( \blacksquare \)

**Theorem 2.5.** Let \( 0 < \beta \leq 1, \gamma \geq 1, p_1 \leq 1, p_2 \leq 1 \) and \( \sigma_i > \tau_i \) for \( i = 1, 2 \).

If the differential inequality

\[
y''(t) \geq \frac{Q_2(t)}{4\gamma^3 / p_1 \gamma^2 / a}y^{\gamma / \alpha}(t - \tau_2 + \sigma_2)
\]

(2.33) has no positive increasing solution, and the differential inequality

\[
y''(t) \geq \frac{Q_1(t)}{p_2^2 / a}y^{\beta / \alpha}(t + \tau_1 - \sigma_1)
\]

(2.34) has no positive decreasing solution, then every solution of equation (1.1) is oscillatory.

**Proof.** Let \( x(t) \) be a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that there exists \( t_1 \geq t_0 \) such that \( x(t) > \).
0, \( x(t - \tau_1) > 0 \), and \( x(t - \sigma_1) > 0 \) for all \( t \geq t_1 \). Setting
\[
z(t) = (x(t) + p_1 x(t - \tau_1) + p_2 x(t + \tau_2))^\alpha,
\]
and
\[
y(t) = z(t) + p_1^\beta z(t - \tau_1) + p_2^\beta x(t + \tau_2).
\]
Then we have \( z(t) > 0, y(t) > 0 \), and
\[
z''(t) = q_1(t) x^\beta(t - \sigma_1) + q_2(t) x^\gamma(t + \sigma_2) \geq 0.
\]
Thus \( z'(t) \) is of one sign eventually. On the other hand
\[
y''(t) = z''(t) + p_1^\beta z''(t - \tau_1) + p_2^\beta z''(t + \tau_2)
= q_1(t) x^\beta(t - \sigma_1) + q_2(t) x^\gamma(t + \sigma_2)
+ p_1^\beta \left[ q_1(t - \tau_1) x^\beta(t - \tau_1 - \sigma_1) + q_2(t - \tau_1) x^\gamma(t - \tau_1 + \sigma_2) \right]
+ p_2^\beta \left[ q_1(t + \tau_2) x^\beta(t + \tau_2 - \sigma_1) + q_2(t + \tau_2) x^\gamma(t + \tau_2 + \sigma_2) \right].
\]
Using the inequality (2.1), we have
\[
q_1(t) x^\beta(t - \sigma_1) + p_1^\beta q_1(t - \tau_1) x^\beta(t - \tau_1 - \sigma_1) + p_2^\beta q_1(t + \tau_2) x^\beta(t + \tau_2 - \sigma_1)
\geq Q_1(t) z^{\beta/\alpha}(t - \sigma_1).
\]
Similarly using the inequality (2.2), and using the fact \( p_1 \leq 1, p_2 \leq 1, 0 < 1 \) and \( \gamma \geq 1 \), we have
\[
q_2(t) x^\gamma(t + \sigma_2) + p_1^\beta q_2(t - \tau_1) x^\gamma(t - \tau_1 + \sigma_2) + p_2^\beta q_2(t + \tau_2) x^\gamma(t + \tau_2 + \sigma_2)
\geq \frac{Q_2(t)}{4^{\gamma-1}} z^{\gamma/\alpha}(t + \sigma_2).
\]
Thus from (2.35), we obtain
\[
y''(t) \geq Q_1(t) z^{\beta/\alpha}(t - \sigma_1) + \frac{Q_2(t)}{4^{\gamma-1}} z^{\gamma/\alpha}(t + \sigma_2).
\]
Now we consider the following two cases:

**Case 1** Assume that $z'(t) > 0$ for all $t \geq t_1$. Then $y'(t) > 0$ for all $t \geq t_1$.

In view of (2.36), we see that

$$y''(t) \geq \frac{Q_2(t)}{4^{1-\gamma}} z^{\gamma/\alpha}(t + \sigma_2).$$

Applying the monotonicity of $z(t)$, we get

$$y(t + \sigma_2) = z(t + \sigma_2) p_1^g z(t - \tau_1 + \sigma_2) + p_2^g z(t + \tau_2 + \sigma_2) \leq p z(t + \tau_2 + \sigma_2).$$

Combining the last two inequalities, we obtain

$$y''(t) \geq \frac{Q_2(t)}{4^{1-\gamma} p^{\gamma/\alpha}} y^{\gamma/\alpha}(t - \tau_2 + \sigma_2).$$

Therefore we see that $y(t)$ is a positive increasing solution of the inequality (2.33), which is a contradiction.

**Case 2** Assume that $z'(t) < 0$ for all $t \geq t_1$. Then $y'(t) < 0$ for all $t \geq t_1$.

In view of (2.36), we see that

$$y''(t) \geq \frac{Q_1(t)}{p^{\beta/\alpha}} y^{\beta/\alpha}(t - \sigma_1).$$

Applying the monotonicity of $z(t)$, we get

$$y(t - \sigma_1) = z(t - \sigma_1) + p_1^g z(t - \tau_1 - \sigma_1) + p_2^g z(t + \tau_2 - \sigma_1) \leq p z(t - \tau_1 - \sigma_1).$$

Combining the last two inequalities, we obtain

$$y''(t) \geq \frac{Q_1(t)}{p^{\beta/\alpha}} y^{\beta/\alpha}(t + \tau_1 - \sigma_1).$$

Therefore, we see that $y(t)$ is a positive decreasing solution of the inequality (2.34), which is a contradiction. This completes the proof of the theorem. ■
From the Lemmas 3 and 4 and the Theorem 2.5 we have the following corollary.

**Corollary 2.6.** Let $\beta < \alpha < \gamma$, $p_1 \leq 1$, $p_2 \leq 1$, $\sigma_1 \geq \tau_1+1$ and $\sigma_2 \geq \tau_2+2$. If
\[
\int_{t=t_0}^{\infty} \left( \int_{s=t}^{t+\sigma_1-\tau_1} Q_1(s) \, ds \right) \, dt = \infty, \tag{2.37}
\]
and
\[
\int_{t=t_0}^{\infty} \left( \int_{s=t-\sigma_2+\tau_2+1}^{t} Q_2(s) \, ds \right) \, dt = \infty, \tag{2.38}
\]
then every solution of equation (1.1) is oscillatory.

**Theorem 2.7.** Let $\beta \geq 1$, $0 < \gamma \leq 1$, $p_1 \geq 1$, $p_2 \geq 1$ and $\sigma_i > \tau_i$ for $i = 1, 2$.

If the differential inequality
\[
y''(t) \geq \frac{Q_2(t)}{p^{\gamma/\alpha}} y^{\gamma/\alpha}(t-\tau_2+\sigma_2) \tag{2.39}
\]
has no positive increasing solution, and the differential inequality
\[
y''(t) \geq \frac{Q_1(t)}{4^{\beta-1}p^{\sigma_1/\alpha}} y^{\sigma_1/\alpha}(t+\tau_1-\sigma_1) \tag{2.40}
\]
has no positive decreasing solution, then every solution of equation (1.1) is oscillatory.

**Proof.** Let $x(t)$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(t-\tau_1) > 0$, and $x(t-\sigma_1) > 0$ for all $t \geq t_1$. By setting
\[
z(t) = [x(t) + p_1 x(t-\tau_1) + p_2 x(t+\tau_2)]^\alpha
\]
and
\[
y(t) = z(t) + p_1^\beta z(t-\tau_1) + p_2^\beta x(t+\tau_2)
\]
it is clear that \(z(t) > 0\), \(y(t) > 0\), and
\[
z''(t) = q_1(t)x^\beta(t - \sigma_1) + q_2(t)x^\gamma(t + \sigma_2).
\]

Therefore \(z'(t)\) is of one sign eventually. On the other hand
\[
y''(t) = z''(t) + p_1^\beta z''(t - \tau_1) + p_2^\beta z''(t + \tau_2)
= q_1(t)x^\beta(t - \sigma_1) + q_2(t)x^\gamma(t + \sigma_2)
\quad + p_1^\beta \left[ q_1(t - \tau_1)x^\beta(t - \tau_1 - \sigma_1) + q_2(t - \tau_1)x^\gamma(t - \tau_1 + \sigma_2) \right]
\quad + p_2^\beta \left[ q_1(t + \tau_2)x^\beta(t + \tau_2 - \sigma_1) + q_2(t + \tau_2)x^\gamma(t + \tau_2 + \sigma_2) \right].
\]

Using the inequality (2.2) and the fact \(p_1 \geq 1\), \(p_2 \geq 1\), \(\beta \geq 1\) and \(0 < \gamma \leq 1\), we get
\[
q_1(t)x^\beta(t - \sigma_1) + p_1^\beta q_1(t - \tau_1)x^\beta(t - \tau_1 - \sigma_1) + p_2^\beta q_1(t + \tau_2)x^\beta(t + \tau_2 - \sigma_1)
\geq \frac{Q_1(t)}{4^{\beta-1}} z^{\beta/\alpha}(t - \sigma_1).
\]

Similarly using the inequality (2.1) and the fact \(p_1 \geq 1, p_2 \geq 1, \beta \geq 1\), and \(0 < \gamma \leq 1\), we get
\[
q_2(t)x^\gamma(t + \sigma_2) + p_1^\gamma q_2(t - \tau_1)x^\gamma(t - \tau_1 + \sigma_2) + p_2^\gamma q_2(t + \tau_2)x^\gamma(t + \tau_2 + \sigma_2)
\geq q_2(t)x^\gamma(t + \sigma_2) + p_1^\gamma q_2(t - \tau_1)x^\gamma(t - \tau_1 + \sigma_2) + p_2^\gamma q_2(t + \tau_2)x^\gamma(t + \tau_2 + \sigma_2)
\geq Q_2(t)z^{\gamma/\alpha}(t + \sigma_2).
\]

Thus from (2.41), we obtain
\[
y''(t) \geq \frac{Q_1(t)}{4^{\beta-1}} z^{\beta/\alpha}(t - \sigma_1) + Q_2(t)z^{\gamma/\alpha}(t + \sigma_2). \tag{2.42}
\]

Now we consider the following two cases:
Case 1 Assume that $z'(t) > 0$ for all $t \geq t_1$. Then $y'(t) > 0$ for all $t \geq t_1$.

In view of (2.42), we see that
\[
y''(t) \geq Q_2(t)z^{\gamma/\alpha}(t + \sigma_2).
\]

Applying the monotonicity of $z(t)$, we get
\[
y(t + \sigma_2) = z(t + \sigma_2)p_1^\beta z(t - \tau_1 + \sigma_2) + p_2^\beta z(t + \tau_2 + \sigma_2) \\
\leq p z(t + \tau_2 + \sigma_2).
\]

Combining the last two inequalities, we obtain
\[
y''(t) \geq Q_2(t) \frac{y^{\gamma/\alpha}(t - \tau_2 + \sigma_2)}{p^{\gamma/\alpha}}.
\]

Therefore $y(t)$ is a positive increasing solution of the inequality (2.39), which is a contradiction.

Case 2 Assume that $z'(t) < 0$ for all $t \geq t_1$. Then $y'(t) < 0$ for all $t \geq t_1$.

In view of (2.42), we see that
\[
y''(t) \geq Q_1(t) \frac{z^{\beta/\alpha}(t - \sigma_1)}{4^{\beta-1} \alpha^2}.
\]

Applying the monotonicity of $z(t)$, we get
\[
y(t - \sigma_1) = z(t - \sigma_1) + p_1^\beta z(t - \tau_1 - \sigma_1) + p_2^\beta z(t + \tau_2 - \sigma_1) \\
\leq p z(t - \tau_1 - \sigma_1).
\]

Combining the last two inequalities, we obtain
\[
y''(t) \geq Q_1(t) \frac{y^{\beta/\alpha}(t + \tau_1 - \sigma_1)}{4^{\beta-1} \alpha^2}.
\]

Therefore, we see that $y(t)$ is a positive decreasing solution of the inequality (2.40), which is a contradiction. This completes the proof of the theorem.
Theorem 2.8. Assume that $\alpha = \beta = \gamma \geq 1$, $\sigma_1 \geq \tau_1$, and $\sigma_2 \geq \tau_2 + 2$. If

$$\limsup_{t \to \infty} \int_{s=t}^{t+\sigma_2-\tau_2-2} (t+\sigma_2-\tau_2-s-1)Q_2(s)ds \geq 4^{\alpha-1} \left(1 + a^\alpha + \frac{b^\alpha}{2^{\alpha-1}}\right), \quad (2.43)$$

and

$$\limsup_{t \to \infty} \int_{s=t-\sigma_1+\tau_1}^{t} (t-s+1)Q_1(s)ds > 4^{\alpha-1} \left(1 + a^\alpha + \frac{b^\alpha}{2^{\alpha-1}}\right) \quad (2.44)$$

hold, then every solution of equation (1.1) is oscillatory.

Proof. Conditions (2.43) and (2.44) imply that the differential inequalities (2.39) and (2.40) have no positive increasing solution and no positive decreasing solution respectively, see [13]. The result now follows from Theorem 2.7. 

Remark 2.1. When $\alpha = \beta = \gamma$, the results obtained here reduced to that of in [14, 19]. Further if, $\alpha = \beta = \gamma = 1$ then the results obtained here complement to that of in [5].

3. Examples. In this section we present some examples to illustrate the main results.

Example 3.1. Consider the differential equation

$$\left(x(t) + \frac{1}{2} x(t-1) + \frac{1}{3} x(t+2)\right)^{''} = \frac{1}{\sqrt{t}} t^{1/3}(t-3) + tx^3(t+4), \quad t \geq 2. \quad (3.1)$$

Here $p_1(t) = \frac{1}{2}$, $p_2(t) = \frac{1}{3}$, $\tau_1 = 1$, $\tau_2 = 2$, $\sigma_1 = 3$, $\sigma_2 = 4$, $\alpha = 1$, $\beta = \frac{1}{3}$, $\gamma = 3$, $q_1(t) = \frac{1}{\sqrt{t}}$, $q_2(t) = t$, $Q_1(t) = \frac{1}{\sqrt{t+1}}$, and $Q_2(t) = t-1$. It is easy to check that all the conditions of Corollary 2.6 are satisfied and hence all the solutions of equation (3.1) are oscillatory.
Example 3.2. Consider the differential equation

\[ ((x(t) + 2x(t-1) + 3x(t+2))^3)^{\mu_2} x^3(t-3) + tx^3(t+5). \]  

(3.2)

Here \( p_1 = 2, p_2 = 3, \tau_1 = 1, \tau_2 = 2, \sigma_1 = 3, \sigma_2 = 5, \alpha = \beta = \gamma = 3, \)
\( q_1(t) = (t+1)^2, q_2(t) = t, Q_1(t) = t^2 \) and \( Q_2(t) = t - 1. \) It is easy to check that all the conditions of Theorem 2.8 are satisfied and hence all the solutions of equation (3.2) are oscillatory.

Remark 3.3. The existing results cannot be applied to Examples (3.1) and (3.2).

References.


[6] J. Dzurina, J. Busa and E. A. Airyan, Oscillation criteria for second-
order differential equations of neutral type with mixed arguments, Diff. Eqns., **38** (2002), 137-140.


[17] Ch. G. Philos, *Oscillation theorems for linear differential equations of


