GLOBAL PROPERTIES OF DECREASING SOLUTIONS OF THE EQUATION \( X'(T) = X(X(T)) + X(T) \) *

SVATOSLAV STANEK †

Abstract. The structure of maximal increasing solutions of the functional differential equation \( x'(t) = x(x(t)) + x(t) \) was completely described in [2]. Here the open problem has been given to describe the structure of maximal decreasing solutions \( A_- \) of this equation. The open problem is considered in our paper. It is proved that the set \( A_- \) is the union of two disjoint nonempty sets \( A^1_- \) and \( A^2_- \) and properties of these sets are considered in detail. Some open problems are stated too.

1. Introduction. The structure of solutions of the functional differential equation

\[
x'(t) = x(x(t)) + x(t)
\]

was considered in [2]. It was proved that (cf. [1]): Any solution \( x \) of (1) is either vanishes identically or \( x' > 0 \) or \( x' < 0 \). Moreover, the global structure of all (strictly) increasing solutions of (1) was precisely studied. The description of the structure of (strictly) decreasing solutions for (1) has been formulated in [2] as an open problem. In this paper we present some results pertain to decreasing solutions of (1).

Observe that global properties of solutions for equation (E): \( x'(t) = x(x(t)) \) were studied by Eder [1] and for equation (W): \( x'(t) = f(x(x(t))) \) by Wang Ke [3], where \( f : \mathbb{R} \to \mathbb{R} \) is continuous increasing or decreasing, \( f(0) = 0 \) and \( |f(z)| \geq \lambda |z| \) for \( z \in \mathbb{R} \) with a positive constant \( \lambda \). The global properties of decreasing solutions of (1) are on principle different from those for both (E) and (W). The main reason consists in the fact that for any decreasing function \( x(t) \) the function \( x(x(t)) \) is increasing (of course, on

---

* Supported by grant no. 201/93/2311 of the Grant Agency of Czech Republic
† Department of Mathematical Analysis and Applications of Mathematics, Faculty of Science, Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic
the set where \( x(x(t)) \) is defined) while the function \( x(x(t)) + x(t) \) has not generally this property. In the end of our paper we give some open problems.

2. Notation, lemmas. In accordance with [3] we introduce the following definitions:

**Definition 1.** We say that \( x \) is a solution of (1) on an interval \( J \) (\( \subset \mathbb{R} \)) if \( x \in C^1(J) \) and (1) is satisfied for \( t \in J \).

**Definition 2.** Let \( x \) and \( y \) be solutions of (1) on intervals \( J \) and \( I \), respectively. We say that \( y \) is a continuation of \( x \) if \( J \subset I \), \( J \neq I \) and \( x(t) = y(t) \) for \( t \in J \). In addition, if \( t \geq s \) (\( t \leq s \)) for each \( t \in I - J \) and each \( s \in J \), then we say that \( y \) is a right (left) continuation of \( x \).

**Definition 3.** We say that \( x \) is a maximal solution of (1) if \( x \) has no continuation, and \( x \) is a right (left) maximal solution of (1) if \( x \) has no right (left) continuation.

**Lemma 1.** ([2]). Let \( x \) be a solution of (1) on an interval \( J \). Then \( x : J \to J \).

**Lemma 2.** ([2]). Let \( x \) be a solution of (1) on an interval \( J \). Then \( \text{sign } x'(t) \) is independent of \( t \) (for \( t \in J \)). Moreover, if \( x'(t_0) = 0 \) for a \( t_0 \in J \), then \( x(t) = 0 \) on \( J \).

Let \( J \subset \mathbb{R} \) be an interval. We write \( A_-(J) \) for the set of all solutions of (1) on the interval \( J \) with sign \( x'(t) = -1 \) on \( J \) (see Lemma 2). \( A_- \) denotes the set of all maximal decreasing solutions of (1).

**Lemma 3.** ([3]). Let \(-\frac{1}{2} < a < 0 \), \( 0 < \varepsilon < \min\{\frac{1}{2}, a + \frac{1}{2}\} \). Then there exists a unique \( x \in A_-([a-\varepsilon, a + \varepsilon]) \) such that \( x(a) = a \).

**Remark 1.** It is easy to verify that \( x_0 \in A_- \), where \( x_0 = -t - 1 \) for \( t \in \mathbb{R} \). Clearly, \( x_0(-\frac{1}{2}) = -\frac{1}{2} \).

**Remark 2.** If \( x \in A_-(J) \), then \( x \in C^\infty(J) \). Moreover, if \( x(a) = a \) for an \( a \in \mathbb{R} \), then \( x'(a) = 2a \), and consequently \( a < 0 \).

**Definition 4.** We say that \( x \in A_-(\mathbb{R}) \) is a derivo-periodic (D-periodic for short) solution of (1) if there exists a \( T > 0 \) such that \( x(t + T) = x(t) - T \) for \( t \in \mathbb{R} \). In this case \( T \) is called a period of the D-periodic solution \( x \) of (1).


**Lemma 4.** Let \( \alpha, \beta \in \mathbb{R}, \alpha < \beta \). Let \( x \in A_-([\alpha, \beta]) \) and \( x(\alpha) = \beta, x(\beta) = \alpha \). Then there exists a D-periodic solution \( y \in A_-(\mathbb{R}) \) with the period \( T = \beta - \alpha \) such that

\[
\text{for } t \in [\alpha, \beta] \text{, } x(t) = y(t).
\]
Proof. First we see that \( x : [\alpha, \beta] \to [\alpha, \beta] \) is bijective and \( x'(\alpha) = \alpha + \beta = x'(\beta) \). Define \( y \in C^1(\mathbb{R}) \) by \( y(t) = x(t - jT) - jT \) for \( t \in [\alpha + jT, \beta + jT] \), where \( j \in \mathbb{Z} \) and \( \mathbb{Z} \) denotes the set of integers. Then \( y(t) = x(t) \) on \([\alpha, \beta]\), \( y(t + kT) = y(t) - kT \) for \( t \in \mathbb{R} \) and \( k \in \mathbb{Z} \). Hence \( y' \) is a \( T \)-periodic function on \( \mathbb{R} \). For \( t \in \mathbb{R} \), \( t = z + jT \) with a \( z \in [\alpha, \beta) \) and a \( j \in \mathbb{Z} \) we have

\[
  y'(t) = y'(z) = x'(z) = x(x(z)) + x(z) = y(y(z)) + y(z)
\]

\[
  = y(y(t - jT)) + y(t - jT) = y(y(t) + jT) + y(t) + jT
\]

\[
  = y(y(t)) + y(t),
\]

which implies \( y \in \mathcal{A}_-(\mathbb{R}) \). Consequently, \( y \) is a \( D \)-periodic solution of \((1)\) with the period \( T \). \( \square \)

REMARK 3. Let assumptions of Lemma 4 be satisfied. Then, by Lemma 4, there exists a \( D \)-periodic solution \( y \in \mathcal{A}_-(\mathbb{R}) \) satisfying \((2)\). We say that \( y \) is a \( D \)-periodic continuation of \( x \).

THEOREM 1. Let \( x \in \mathcal{A}_-(J) \) be a maximal solution. Then \( J = \mathbb{R} \).

Proof. Assume \( J \neq \bar{J} \), where \( \bar{J} \) is the closure (in \( \mathbb{R} \)) of \( J \). Let \( J \) be a bounded interval with end points \( t_0, t_1, t_0 < t_1 \). Then there exist the finite limits

\[
  \lim_{t \to t_0^+} x(t) = B, \quad \lim_{t \to t_1^-} x(t) = A,
\]

and \([A, B] \subseteq \bar{J}\) since \( x \) is decreasing on \( J \) and \( x : J \to J \) by Lemma 1. We recall that \( B = x(t_0) \) provided \( t_0 \in J \) and \( A = x(t_1) \) provided \( t_1 \in J \). Define \( y \in C^0(\bar{J}) \) by

\[
  y(t) = \begin{cases} 
    B & \text{for } t = t_0 \\
    x(t) & \text{for } t \in (t_0, t_1) \\
    A & \text{for } t = t_1.
  \end{cases}
\]

Using the equality \((c \in J)\)

\[
  x(t) = e^t x(c)e^{-c} + \int_c^t e^{-s} x(x(s)) ds, \quad t \in J,
\]

which follows from the equality \( x'(t) = x(x(t)) + x(t) \) on \( J \), we see that \( y \) satisfies the equality

\[
  y(t) = e^t (y(c)e^{-c} + \int_c^t e^{-s} y(x(s)) ds), \quad t \in \bar{J}.
\]

Hence \( y \in \mathcal{A}_-(\bar{J}) \) and \( y \) is a continuation of \( x \), a contradiction.
Let \( J = (-\infty, t_2) \) with a \( t_2 \in \mathbb{R} \). Since \( (x(x(t)))' = x'(x(t))x'(t) > 0 \) on \( J \), \( x(x(t)) \) is increasing on the same interval, and consequently \( x(x(t)) \geq x(x(c)) \) on \([c, t_2)\) for each \( c \in J \). Hence (cf. (3))

\[
x(t) \geq e^t \left( x(c)e^{-c} + x(x(c)) \int_c^t e^{-s} ds \right)
\]

for \( t \in [c, t_2) \), and so there exists the finite limit

\[
\lim_{t \to t_2^-} x(t) = C.
\]

Set

\[
z(t) = \begin{cases} x(t) & \text{for } t \in J \\ C & \text{for } t = t_2. \end{cases}
\]

Then \( z \) is a right continuation of \( x \), a contradiction.

Let \( J = (t_3, \infty) \) with a \( t_3 \in \mathbb{R} \). By Lemma 1, \( x(t) > t_3 \) on \( J \), and consequently there exists the finite limit

\[
\lim_{t \to t_3^+} x(t) = D.
\]

Then the function \( w : [t_3, \infty) \to \mathbb{R} \),

\[
w(t) = \begin{cases} D & \text{for } t = t_3 \\ x(t) & \text{for } t \in J \end{cases}
\]

is a left continuation of \( x \), a contradiction.

This proves that \( J = \bar{J} \) and then either \( J = \mathbb{R} \) or \( J = [\alpha, \beta] \) or \( J = (-\infty, \gamma) \) or \( J = [\delta, \infty) \) with some \( \alpha, \beta, \gamma, \delta \in \mathbb{R}, \alpha < \beta \). We now prove that \( J = \mathbb{R} \).

Assume \( J = [\alpha, \beta] \). If \( x(t_0) \in (\alpha, \beta) \) with \( t_0 = \alpha \) or \( \beta \), then the Cauchy initial problem

\[
y' = x(y) + y, \quad y(t_0) = x(t_0)
\]

has a unique solution \( y(t) \) on a neighbourhood \( U \) of the point \( t = t_0 \) since \( x \in C^1(J) \); hence the function \( r : J \cup U \to J \),

\[
r(t) = \begin{cases} x(t) & \text{for } t \in J \\ y(t) & \text{for } t \in U - J \end{cases}
\]
is a continuation of \( x \), a contradiction. Thus \( x(\alpha) = \beta, x(\beta) = \alpha \) and then, by Lemma 4, there exists a \( D \)-periodic continuation of \( x \), a contradiction.

Let \( J = (-\infty, \gamma) \). If \( x(\gamma) \in (-\infty, \gamma) \), there exists a right continuation of \( x \); hence \( x(\gamma) = \gamma \). Since \( x \) is decreasing, \( x(t) > \gamma \) on \( (-\infty, \gamma) \), which contradicts Lemma 1.

Let \( J = [\delta, \infty) \). If \( x(\delta) > \delta \), there exists a left continuation of \( x \); hence \( x(\delta) = \delta \). Then \( x(t) < \delta \) on \( (\delta, \infty) \), which again contradicts Lemma 1. \( \square \)

From Theorem 1 immediately follows:

**Corollary 1.** \( \mathcal{A}_- = \mathcal{A}_-(\mathbb{R}) \).

**Corollary 2.** Let \( x \in \mathcal{A}_- \). Then there exists a unique \( a \in \mathbb{R} \) such that \( x(a) = a \). Moreover, \( a < 0 \).

**Proof.** Since \( (x(t) - t)' = x'(t) - 1 < -1 \) on \( \mathbb{R} \), \( \lim_{t \to -\infty} (x(t) - t) = \infty \) and \( \lim_{t \to -\infty} (x(t) - t) = -\infty \), and consequently there exists a unique solution of the equation \( x(t) - t = 0 \), say \( a \). Then \( x(a) = a \) and, by Remark 2, \( a < 0 \). \( \square \)

**Lemma 5.** Let \( x \in \mathcal{A}_- \). Then either

\[
\lim_{t \to -\infty} x(t) = \infty, \quad \lim_{t \to \infty} x(t) = -\infty
\]

or

\[
\lim_{t \to -\infty} x(t) < \infty, \quad \lim_{t \to \infty} x(t) = -\infty.
\]

**Proof.** Assume, on the contrary, that \( x \) is bounded below on \( \mathbb{R} \); that is, there exists the finite limit

\[
\lim_{t \to -\infty} x(t) = B.
\]

Since \( e^t \to \infty \) as \( t \to \infty \), we deduce from (3) (with \( J = \mathbb{R} \)) that

\[
\lim_{t \to -\infty} \left( x(c)e^{-c} + \int_c^t e^{-s}x(x(s)) \, ds \right) = 0
\]

and then, by the L'Hospital rule,

\[
B = \lim_{t \to -\infty} x(t) = \lim_{t \to -\infty} \frac{x(c)e^{-c} + \int_c^t e^{-s}x(x(s)) \, ds}{e^{-t}} = -\lim_{t \to -\infty} x(t) = -x(B).
\]

Using the equality \( x(B) = -B \) and the inequality \( x(t) > B \) for \( t \in \mathbb{R} \), which follows from the fact that \( x \) is decreasing, we see that \( 0 > x'(B) = x(x(B)) + x(B) = x(-B) - B > B - B = 0 \), a contradiction.
Thus
\[\lim_{t \to \infty} x(t) = -\infty,\]
which proves the lemma.

Using the dichotomy result in Lemma 5, we introduce the following notation, which is useful for a better consideration of global properties of solutions.

Set
\[A_1^1 = \{ x \in A_-, \lim_{t \to -\infty} x(t) < \infty \},\]
\[A_1^2 = \{ x \in A_-, \lim_{t \to -\infty} x(t) = \infty \}.\]

**Remark 4.** \(A_1^1 \cap A_1^2 = \emptyset\) and, by Lemma 5, \(A_- = A_1^1 \cup A_1^2\). Since (cf. Remark 1) \(-t - 1 \in A_1^2\), \(A_1^2\) is a nonempty set.

**Lemma 6.** Let \(x \in A_1^1\) and \(\lim_{t \to -\infty} x(t) = A\). Then
(i) \(A > 0\), \(x(A) = -A\)
and
(ii) \(\lim_{t \to -\infty} x'(t) = 0\).

**Proof.** Using equality (3) (with \(J = \mathbb{R}\)) and the L'Hospital rule, we obtain
\[A = \lim_{t \to -\infty} x(t) = \lim_{t \to -\infty} \frac{x(c)e^{-c} + \int_0^t e^{-s}x(x(s))ds}{e^{-t}} = -\lim_{t \to -\infty} x(x(t)) = -x(A).\]
Hence \(x(A) = -A\) and since \(A > x(t)\) on \(\mathbb{R}\), \(-A = x(A) < A\), which implies \(A > 0\). Then \(\lim_{t \to -\infty} x'(t) = \lim_{t \to -\infty}(x(x(t)) + x(t)) = x(A) + A = 0\).

**Corollary 3.** Let \(x \in A_-\). Then \(x \in A_2^2\) if and only if \(x(t) < -t\) for \(t \in \mathbb{R}\).

**Proof.** Let \(x \in A_2^2\). Then \(x\) maps \(\mathbb{R}\) onto \(\mathbb{R}\) and from the relation \(0 > x'(t) = x(x(t)) + x(t)\) we deduce \(x(t) + t < 0\) on \(\mathbb{R}\). On the other hand if \(x \in A_-\) and \(x(t) < -t\) on \(\mathbb{R}\), then necessarily \(x \in A_2^2\) by Lemma 6 (see (i)) and Remark 4.

**Remark 5.** Let \(x \in A_-\), \(x(a) = a\) (see Corollary 2) and \(x(x(\alpha)) = \alpha\) for some \(\alpha \in \mathbb{R}\), \(\alpha \neq a\). Set \(\beta = x(\alpha)\). Then \(x(x(\beta)) = \beta, x(\beta) = \alpha, x'(\alpha) = \alpha + \beta = x'(\beta)\) and \(\alpha \neq \beta\). If for example \(\alpha < \beta\), then \(\alpha < a < \beta\).

Indeed, if \(\alpha > a\), then \(\beta = x(\alpha) < x(a) = a\), a contradiction. Thus \(\alpha < a\) and then \(\beta = x(\alpha) > x(a) = a\).
Lemma 7. Let $x \in A_-$ and $x(\alpha) \neq \alpha$, $x(x(\alpha)) = \alpha$, $(x(x(t)))'_{t=\alpha} = 1$ for an $\alpha \in \mathbb{R}$. Then

$$x(t) = -t - 1 \quad \text{for } t \in \mathbb{R}.$$  

Proof. Set $\beta = x(\alpha)$ and let (cf. Corollary 2) $x(\alpha) = a$ for an $a \in (-\infty, 0)$. Then $\alpha \neq a$, $\alpha \neq \beta$ and (cf. Remark 5) $x(\beta) = \beta$, $x(\alpha) = \beta$, $\min\{\alpha, \beta\} < a < \max\{\alpha, \beta\}$, $x'(\alpha) = \alpha + \beta = x'(\beta)$, and consequently

$$\alpha + \beta = -1$$  

since $x' < 0$ and $1 = (x(x(t)))'_{t=\alpha} = x'(\alpha)x'(\beta) = (\alpha + \beta)^2$. Therefore (4) is satisfied for $t \in \{\alpha, \beta\}$. Without loss of generality we can assume $\alpha < \beta$. We now prove that $x(t) = -t - 1$ for $t \in [\alpha, \beta]$. Assume, on the contrary, that

$$x(t) \neq -t - 1 \quad \text{for } t \in [\alpha, \beta].$$  

Let $x(t) = -t - 1$ for $t \in [\alpha, \alpha_1]$ with an $\alpha_1 \in (\alpha, a)$. Then $(x'(t)) = -1 = x(x(t)) - t - 1$ for $t \in [\alpha, \alpha_1]$, which implies $x(x(t)) = t$ on this interval, and therefore $x'(t) = -1$ for $t \in [x(\alpha_1), x(\alpha)] = [x(\alpha_1), \beta]$. Hence $x(t) = -t - 1$ on $[x(\alpha_1), \beta]$ since $x(\beta) = \alpha$ and (5) holds. Similarly for $x(t) = -t - 1$ on $[\beta_1, \beta]$ with a $\beta_1 \in (a, \beta)$. This proves that there exist (cf. (6)) $\alpha_2, \beta_2 \in [\alpha, \beta]$, $\alpha_2 < a < \beta_2$ such that $x(t) = -t - 1$ for $t \in [\alpha, \alpha_2] \cup [\beta_2, \beta]$ and

$$\max\{|x(t) + t + 1|; \alpha_2 \leq t \leq \alpha_2 + \varepsilon\} > 0,$$

$$\max\{|x(t) + t + 1|; \beta_2 - \varepsilon \leq t \leq \beta_2\} > 0$$

for $\varepsilon \in (0, \beta_2 - \alpha_2]$. Then for any $\alpha_3, \beta_3$, $\alpha_2 < \alpha_3 < a < \beta_3 < \beta_2$ such that $x(\alpha_3) \neq -\alpha_3 - 1$, $x(\beta_3) \neq -\beta_3 - 1$, we have $x(\alpha_3) < \beta_2$, $x(\beta_3) > \alpha_2$, which imply $x(\alpha_2) \leq \beta_2$, $x(\beta_2) \geq \alpha_2$. On the other hand, from the properties of the numbers $\alpha_2$ and $\beta_2$ we see that $x(\alpha_2) \geq \beta_2$, $x(\beta_2) \leq \alpha_2$. Consequently,

$$x(\alpha_2) = \beta_2, \quad x(\beta_2) = \alpha_2, \quad \beta_2 = -\alpha_2 - 1.$$  

Set $y(t) = -t - 1$ for $t \in \mathbb{R}$ and

$$M(\varepsilon) = \max\{|x(t) - y(t)|; \alpha_2 \leq t \leq \alpha_2 + \varepsilon\},$$

$$N(\varepsilon) = \max\{|x(t) - y(t)|; \beta_2 - \varepsilon \leq t \leq \beta_2\}$$

for $\varepsilon \in [0, \beta_2 - \alpha_2]$. Then $M, N$ are continuous nondecreasing on $[0, \beta_2 - \alpha_2]$, $M(0) = 0 = N(0)$ and (cf. (7)) $M(\varepsilon) > 0$, $N(\varepsilon) > 0$ for $\varepsilon \in (0, \beta_2 - \alpha_2]$. By
the Taylor formula,
\begin{equation}
x(x(t)) - y(y(t)) = \left( x(x(t)) - x(y(t)) \right) + \left( x(y(t)) - y(y(t)) \right)
= x'(\rho)(x(t) - y(t)) + \left( x(y(t)) - y(y(t)) \right),
\end{equation}
where \( \rho \) lies between \( x(t) \) and \( y(t) \); hence
\begin{equation}
|x(x(t)) - y(y(t))| \leq L|x(t) - y(t)| + |x(y(t)) - y(y(t))|
\end{equation}
for \( t \in [\alpha_2, \beta_2] \), where \( L = \max \{|x'(t)|; \alpha_2 \leq t \leq \beta_2 \} \). Using the equalities (for \( t \in \mathbb{R} \))
\begin{align*}
x(t) - y(t) &= \int_{\alpha_2}^{t} (x(s) - y(s)) \, ds + \int_{\alpha_2}^{t} (x(x(s)) - y(y(s))) \, ds, \\
x(t) - y(t) &= \int_{\beta_2}^{t} (x(s) - y(s)) \, ds + \int_{\beta_2}^{t} (x(x(s)) - y(y(s))) \, ds
\end{align*}
and (10), we obtain
\begin{align*}
|x(t) - y(t)| &\leq (1 + L) \int_{\alpha_2}^{t} |x(s) - y(s)| \, ds + \int_{\alpha_2}^{t} |x(y(s)) - y(y(s))| \, ds \\
&= (1 + L) \int_{\alpha_2}^{t} |x(s) - y(s)| \, ds + \int_{-\alpha_2-1}^{t} |x(s) - y(s)| \, ds, \\
|x(t) - y(t)| &\leq (1 + L) \int_{t}^{\beta_2} |x(s) - y(s)| \, ds + \int_{t}^{\beta_2} |x(y(s)) - y(y(s))| \, ds \\
&= (1 + L) \int_{t}^{\beta_2} |x(s) - y(s)| \, ds + \int_{-t-1}^{-\beta_2-1} |x(s) - y(s)| \, ds
\end{align*}
for \( t \in [\alpha_2, \beta_2] \). Then (cf. (8))
\begin{align*}
(11) \quad |x(t) - y(t)| &\leq (1 + L)(t - \alpha_2)M(t - \alpha_2) + (t - \alpha_2)N(t - \alpha_2), \\
(12) \quad |x(t) - y(t)| &\leq (1 + L)(\beta_2 - t)N(\beta_2 - t) + (\beta_2 - t)M(\beta_2 - t)
\end{align*}
for \( t \in [\alpha_2, \beta_2] \). Fix \( \varepsilon \in [0, \beta_2 - \alpha_2] \). By (11) and (12),
\begin{align*}
|x(t) - y(t)| &\leq (1 + L)\varepsilon M(\varepsilon) + \varepsilon N(\varepsilon) \quad \text{for } t \in [\alpha_2, \alpha_2 + \varepsilon], \\
|x(t) - y(t)| &\leq (1 + L)\varepsilon N(\varepsilon) + \varepsilon M(\varepsilon) \quad \text{for } t \in [\beta_2 - \varepsilon, \beta_2],
\end{align*}
and therefore
\[ M(\varepsilon) \leq (1 + L)\varepsilon M(\varepsilon) + \varepsilon N(\varepsilon), \quad N(\varepsilon) \leq (1 + L)\varepsilon N(\varepsilon) + \varepsilon M(\varepsilon) \]

for each \( \varepsilon \in [0, \beta_2 - \alpha_2] \). Hence
\[ M(\varepsilon) + N(\varepsilon) \leq \varepsilon(2 + L)(M(\varepsilon) + N(\varepsilon)) \]

and since \( M(\varepsilon) + N(\varepsilon) > 0 \) on \( (0, \beta_2 - \alpha_2) \), we have \( 1 \leq \varepsilon(2 + L) \) for each \( \varepsilon \in (0, \beta_2 - \alpha_2) \), a contradiction. Consequently, \( x(t) = -t - 1 \) for \( t \in [\alpha, \beta] \) and \( \alpha = -\frac{1}{2} \).

Assume \( x(t) \neq -t - 1 \) on \( \mathbb{R} \) and set
\[
\alpha^* = \inf \{ t; t \leq \alpha, \ x(s) = -s - 1 \ \text{for} \ s \in [t, \beta] \},
\]
\[
\beta^* = \sup \{ t; t \geq \beta, \ x(s) = -s - 1 \ \text{for} \ s \in [\alpha, t] \}.
\]

Clearly, \( -\infty \leq \alpha^* \leq \alpha < a < \beta \leq \beta^* \leq \infty \). If \( \alpha^* = -\infty \), then \( x(t) = -t - 1 \) on \( (-\infty, \beta^*] \), and consequently \( -1 = x'(t) = -t - 1 + x(x(t)) \) on \( (-\infty, \beta^*] \); hence \( x(x(t)) = t \) for \( t \in (-\infty, \beta^*] \) and then \( x'(t) = -1 \) on \( [a, \infty) \) since \( 1 = x'(x(t))x'(t) = -x'(x(t)) \) for \( t \in (-\infty, \beta^*] \), \( x(t) \geq a \) for \( t \in (-\infty, a] \subset (-\infty, \beta^*] \) and \( \lim_{t \to -\infty} x(t) = \infty \). Equalities \( x(\beta) = -\beta - 1 \) and \( x'(t) = -1 \) for \( t \in [a, \infty) \) imply \( x(t) = -t - 1 \) on \( [a, \infty) \); hence \( x(t) = -t - 1 \) for \( t \in \mathbb{R} \), a contradiction. Thus \( \alpha^* > -\infty \). Similarly for \( \beta^* < \infty \). We see that \( [\alpha^*, \beta^*] \) is the maximal interval containing \( a \left( = -\frac{1}{2} \right) \) such that \( x(t) = -t - 1 \) on \( [\alpha^*, \beta^*] \). Since \( -1 = x'(t) = -t - 1 + x(x(t)) \) for \( t \in [\alpha^*, \beta^*] \), \( x(x(t)) = t \) on this interval, and consequently \( x(-t - 1) = t \) for \( t \in [\alpha^*, \beta^*] \). Hence
\[(13) \quad x(-\alpha^* - 1) = \alpha^*, \quad x(-\beta^* - 1) = \beta^* \]

and \( x(t) = -t - 1 \) for \( t \in [-\beta^* - 1, -\alpha^* - 1] \). Thus \( (-\frac{1}{2}) \in [-\beta^* - 1, -\alpha^* - 1] \subset [\alpha^*, \beta^*] \), which occurs if and only if \( \alpha^* + \beta^* = -1 \). Then (cf. (13))
\[ x(\beta^*) = \alpha^*, \quad x(\alpha^*) = \beta^*. \]

Let \( A, B : [0, \infty) \to [0, \infty) \),
\[ A(\varepsilon) = \max \{ |x(t) + t + 1|; \ \alpha^* - \varepsilon \leq t \leq \alpha^* \}, \]
\[ B(\varepsilon) = \max \{ |x(t) + t + 1|; \ \beta^* \leq t \leq \beta^* + \varepsilon \} \]

and
\[ K = \max \{ \max \{ |x'(t)|; \ \alpha^* \geq t \geq \min \{ \alpha^* - 1, x(\beta^* + 1) \} \}, \max \{ |x'(t)|; \ \beta^* \leq t \leq \max \{ \beta^* + 1, x(\alpha^* - 1) \} \} \}. \]
Then $A, B$ are continuous nondecreasing on $[0, \infty)$, $A(0) = 0 = B(0)$ and $A(\varepsilon) > 0$, $B(\varepsilon) > 0$ for $\varepsilon > 0$. Since $x(\alpha*) = y(\alpha*)$, $x(\beta*) = y(\beta*)$ with $y(t) = -t - 1$ on $\mathbb{R}$,

\[(14) \quad x(t) - y(t) = \int_{\alpha*}^{t} (x(s) - y(s)) \, ds + \int_{\alpha*}^{t} (x(x(s)) - y(y(s))) \, ds,\]
\[(15) \quad x(t) - y(t) = \int_{\beta*}^{t} (x(s) - y(s)) \, ds + \int_{\beta*}^{t} (x(x(s)) - y(y(s))) \, ds\]

for $t \in \mathbb{R}$. Using equality (9), where $\rho$ lies between $x(t)$ and $y(t)$, we get

\[(16) \quad |x(x(t)) - y(y(t))| \leq K|x(t) - y(t)| + |x(y(t)) - y(y(t))|\]

for $t \in [\alpha* - 1, \alpha*] \cup [\beta*, \beta* + 1]$. Then (cf. (14) and (16))

\[(17) \quad |x(t) - y(t)| \leq (1 + K) \int_{t}^{\alpha*} |x(s) - y(s)| \, ds + \int_{t}^{\alpha*} |x(y(s)) - y(y(s))| \, ds\]

for $t \in [\alpha* - 1, \alpha*]$ and similarly (cf. (15) and (16))

\[(18) \quad |x(t) - y(t)| \leq (1 + K) \int_{\beta*}^{t} |x(s) - y(s)| \, ds + \int_{\beta*}^{t} |x(s) - y(s)| \, ds\]

for $t \in [\beta*, \beta* + 1]$. Therefore (cf. (17) and (18))

\[(19) \quad |x(t) - y(t)| \leq (1 + K)(\alpha* - t)A(\alpha* - t) + (\alpha* - t)B(\alpha* - t)\]

for $t \in [\alpha* - 1, \alpha*]$,

\[(20) \quad |x(t) - y(t)| \leq (1 + K)(t - \beta*)B(t - \beta*) + (t - \beta*)A(t - \beta*)\]

for $t \in [\beta*, \beta* + 1]$ since $\alpha* + \beta* = -1$. From (19) and (20),

\[|x(t) - y(t)| \leq (1 + K)\varepsilon A(\varepsilon) + \varepsilon B(\varepsilon) \quad \text{for } \alpha* - 1 \leq \alpha* - \varepsilon \leq t \leq \alpha*\]

and

\[|x(t) - y(t)| \leq (1 + K)\varepsilon B(\varepsilon) + \varepsilon A(\varepsilon) \quad \text{for } \beta* \leq t \leq \beta* + \varepsilon \leq \beta* + 1.\]

Then

\[A(\varepsilon) \leq \varepsilon [(1 + K)A(\varepsilon) + B(\varepsilon)], \quad B(\varepsilon) \leq \varepsilon [(1 + K)B(\varepsilon) + A(\varepsilon)],\]
and consequently $A(\varepsilon) + B(\varepsilon) \leq \varepsilon(2 + K)(A(\varepsilon) + B(\varepsilon))$ for $\varepsilon \in [0, 1]$. Since $A(\varepsilon) + B(\varepsilon) > 0$ on $(0, 1]$, $1 \leq \varepsilon(2 + K)$ for each $\varepsilon \in (0, 1]$, a contradiction. Hence $x(t) = -t - 1$ for $t \in \mathbb{R}$, and the proof is complete.

**Corollary 4.** Let $x \in A_-$, $x(t) \neq -t - 1$ on $\mathbb{R}$. Then $(x(x(t)))'_{t=\alpha} \neq 1$ for each $\alpha \in \mathbb{R}$ such that $x(\alpha) \neq \alpha$ and $x(x(\alpha)) = \alpha$.

**Lemma 8.** Let $x \in A_1$ and $x(a) = a$ for an $a \in \mathbb{R}$. Then $a \in (-\frac{1}{2}, 0)$,

$$x(x(t)) > t \quad \text{for } t < a$$

and

$$x(x(t)) < t \quad \text{for } t > a.$$

**Proof.** First we see that $x(x(t)) \neq t$ for $t > a$ implies (cf. Remark 5) $x(x(t)) \neq t$ for $t < a$. Assume, on the contrary, that $x(x(\beta_1)) = \beta_1$ for a $\beta_1 > a$. Then for $\alpha_1 = x(\beta_1)$ we have $x(x(\alpha_1)) = \alpha_1$, $x(\alpha_1) = \beta_1$ and $\alpha_1 < a$. Let $A = \lim_{t \to -\infty} x(t)$. By Lemma 5 and Lemma 6, the increasing function $x(x(t))$ maps $\mathbb{R}$ onto $(-A, A)$ and therefore there exists a $\beta \geq \beta_1$ such that $x(x(\beta)) = \beta$, $x(x(t)) < t$ for $t > \beta$, and consequently $x(x(\alpha)) = \alpha$ and $x(x(t)) \neq t$ for $t < \alpha$ with $\alpha = x(\beta)$. Since $(x(x(t)))'_{t=\alpha} = (x(x(t)))'_{t=\beta} \leq 1,$

$$(x(x(t)))'_{t=\alpha} \neq 1 \text{ by Corollary 4;} \text{ hence } x(x(t)) < t \text{ for } t < \alpha.$$ Moreover, by Lemma 4, there exists a $D$-periodic solution $y(t)$ of (1) such that $x(t) = y(t)$ for $t \in [\alpha, \beta]$. We now show that

$$(21') \quad \max \{|x(s) - y(s)|; \quad t \leq s \leq \alpha\} > 0 \quad \text{for } t < \alpha$$

and

$$(21'') \quad \max \{|x(s) - y(s)|; \quad \beta \leq s \leq t\} > 0 \quad \text{for } t > \beta.$$
for \( t \in [\alpha_2 - \varepsilon_0, \alpha_2] \) since \( y(s) \in [\alpha_2, \beta_2] \), \( x(y(s)) = y(y(s)) \) for \( s \in [\alpha_2 - \varepsilon_0, \alpha_2] \).

Setting \( m = \max \{|x'(t)|; \alpha_2 \leq t \leq \max \{\beta_2, x(\alpha_2 - \varepsilon_0)\}\} \), then

\[
|x(x(s)) - x(y(s))| = |x'(\rho)||x(s) - y(s)| \leq m|x(s) - y(s)|
\]

for \( s \in [\alpha_2 - \varepsilon_0, \alpha_2] \), by the Taylor formula, where \( \rho \) lies between \( x(s) \) and \( y(s) \). From (22) and (23) we obtain

\[
|x(t) - y(t)| \leq (1 + m) \int_t^{\alpha_2} |x(s) - y(s)| ds, \quad t \in [\alpha_2 - \varepsilon_0, \alpha_2]
\]

and then \( x(t) = y(t) \) on \( [\alpha_2 - \varepsilon_0, \alpha_2] \) by the Gronwall lemma, a contradiction. Similarly for \( y(\beta_2) > \alpha_2 \). Thus \( x(t) = y(t) \) on \( [\alpha_2, \beta_2] \) and \( x(\alpha_2) = \beta_2 \), \( x(\beta_2) = \alpha_2 \); hence \( x(x(\alpha_2)) = \alpha_2 \), \( x(x(\beta_2)) = \beta_2 \) and then \( \alpha_2, \beta_2] \subset [\alpha, \beta] \), a contradiction. We see that (21') and (21'') hold. Set

\[
p(\mu) = \max \{|x(t) - y(t)|; \alpha - \mu \leq t \leq \alpha\},
q(\mu) = \max \{|x(t) - y(t)|; \beta \leq t \leq \beta + \mu\}
\]

for \( \mu \in [0, \infty) \). Then \( p, q \in C^0([0, \infty)) \), \( p(0) = 0 = q(0) \) and (cf. (21') and (21'')) \( p(\mu) > 0, q(\mu) > 0 \) for \( \mu > 0 \). Since \( x'(\alpha) = y'(\alpha) = x'(\beta) = y'(\beta) = \alpha + \beta \) \((< 0)\) and \( 1 > (x(x(t)))'_{t=\alpha} = (\alpha + \beta)^2 \), we conclude that \( \alpha + \beta > -1 \) and therefore there exists an \( \varepsilon > 0 \) such that

\[
x'(t) \leq 1, \quad x(t) \leq -t + \alpha + \beta, \quad y(t) \leq -t + \alpha + \beta \quad \text{for } t \in [\alpha - \varepsilon, \alpha]
\]

and

\[
x'(t) \leq 1, \quad x(t) \geq -t + \alpha + \beta, \quad y(t) \geq -t + \alpha + \beta \quad \text{for } t \in [\beta, \beta + \varepsilon].
\]

Thus

\[
|x(x(t)) - y(y(t))| \leq |x'(\rho)||x(t) - y(t)| + |x(y(t)) - y(y(t))|
\]

\[
\leq |x(t) - y(t)| + |x(y(t)) - y(y(t))|
\]

for \( t \in [\alpha - \varepsilon, \alpha] \cup [\beta, \beta + \varepsilon] \), where \( \rho \) lies between \( x(t) \) and \( y(t) \). Set

\[
M = \max \{|(y^{-1}(t))'|; t \in [\alpha - \varepsilon, \alpha] \cup [\beta, \beta + \varepsilon]\},
\]

where \( y^{-1} \) denotes the inverse function to \( y \) (on \( \mathbb{R} \)). From (24) and the definition of \( M \) we deduce that

\[
|x(t) - y(t)| \leq \int_t^{\alpha} |x(s) - y(s)| ds + \int_t^{\alpha} |x(x(s)) - y(y(s))| ds
\]

\[
\leq 2 \int_t^{\alpha} |x(s) - y(s)| ds + \int_t^{\alpha} |x(y(s)) - y(y(s))| ds
\]

\[
= 2 \int_t^{\alpha} |x(s) - y(s)| ds + \int_{\beta}^{y(t)} |x(s) - y(s)||y^{-1}(s)'| ds
\]

\[
\leq 2 \int_t^{\alpha} |x(s) - y(s)| ds + M \int_{\beta}^{\gamma(t) + \alpha + \beta} |x(s) - y(s)| ds,
\]
for $t \in [\alpha - \varepsilon, \alpha]$. Then

$$|x(t) - y(t)| \leq 2(\alpha - t)p(\alpha - t) + M(\alpha - t)q(\alpha - t), \quad t \in [\alpha - \varepsilon, \alpha]$$

and

$$|x(t) - y(t)| \leq 2\mu p(\mu) + M\mu q(\mu) \quad \text{for} \quad \alpha - \varepsilon \leq \alpha - \mu \leq t \leq \alpha,$$

which implies

$$p(\mu) \leq 2\mu p(\mu) + M\mu q(\mu), \quad 0 \leq \mu \leq \varepsilon. \quad (26)$$

Similarly,

$$|x(t) - y(t)| \leq 2 \int_{\beta}^{t} |x(s) - y(s)| \, ds + \int_{y(t)}^{\alpha} |x(s) - y(s)||y^{-1}(s)'| \, ds$$

$$\leq 2 \int_{\beta}^{t} |x(s) - y(s)| \, ds + M \int_{-t + \alpha + \beta}^{\alpha} |x(s) - y(s)| \, ds$$

for $t \in [\beta, \beta + \varepsilon]$. Then

$$|x(t) - y(t)| \leq 2(t - \beta)q(t - \beta) + M(t - \beta)p(t - \beta), \quad t \in [\beta, \beta + \varepsilon],$$

and consequently

$$q(\mu) \leq 2\mu q(\mu) + M\mu p(\mu), \quad 0 \leq \mu \leq \varepsilon. \quad (27)$$

From (26) and (27) it follows that

$$p(\mu) + q(\mu) \leq \mu(2 + M)(p(\mu) + q(\mu)), \quad 0 \leq \mu \leq \varepsilon,$$

and since $p(\mu) + q(\mu) > 0$ for $\mu > 0$, $1 \leq \mu(2 + M)$ for each $\mu \in (0, \varepsilon]$, a contradiction. We have proved that $x(x(t)) > t$ for $t < a$ and $x(x(t)) < t$ for $t > a$.

It remains to prove that $a \in (-\frac{1}{2}, 0)$. We know that $(0 >) x'(a) = 2a$; hence $4a^2 = (x(x(t)))'_{t=a} \leq 1$ and so $a \in [-\frac{1}{2}, 0)$. Assume $a = -\frac{1}{2}$. Since $z(t) = -t - 1$ is a solution of (1) (on $\mathbb{R}$) and $z(-\frac{1}{2}) = -\frac{1}{2}$, we have

$$x(t) - z(t) = e^{t} \int_{-\frac{1}{2}}^{t} e^{-s}(x(x(s)) - z(s)) \, ds = e^{t} \int_{-\frac{1}{2}}^{t} e^{-s}(x(x(s)) - s) \, ds$$

for $t \in \mathbb{R}$. Then the inequalities $x(x(t)) > t$ on $(-\infty, -\frac{1}{2})$ and $x(x(t)) < t$ on $(-\frac{1}{2}, \infty)$ imply $x(t) - z(t) < 0$ on $\mathbb{R} \setminus \{-\frac{1}{2}\}$; hence for $A = \lim_{t \to -\infty} x(t)$ we
have \( x(A) < z(A) = -A - 1 \), which contradicts \( x(A) = -A \) (see Lemma 6). Consequently, \( a \in (-\frac{1}{2}, 0) \).

**Lemma 9.** Let \( x \in A_- \), \( x(a) = a \) for an \( a \in (-\infty, 0) \) and let \( x(x(t)) > t \) for \( t < a \) and \( x(x(t)) < t \) for \( t > a \). Then \( x \in A_1 \) and \( a \in (-\frac{1}{2}, 0) \).

**Proof.** Assume the result is not true. Then \( x \in A_2 \) and therefore \( x \) maps \( \mathbb{R} \) onto itself and (cf. Corollary 3) \( x(t) < -t \) on \( \mathbb{R} \). For \( t < b \leq a \) (cf. (3) with \( c = b \),

\[
x(t) = e^t \left( x(b) e^{-b} - \int_t^b e^{-s} x(s) \, ds \right) < e^t \left( x(b) e^{-b} - \int_t^b e^{-s} \, ds \right) = (x(b) + b + 1) e^{-b} - t - 1
\]

and

\[
(28) \quad x(t) + t + 1 < (x(b) + b + 1) e^{-b} \quad \text{for } t < b \leq a.
\]

Similarly we can verify that

\[
(29) \quad x(t) + t + 1 < (x(c) + c + 1) e^{-c} \quad \text{for } a < c < t.
\]

If \( x(T_0) + T_0 + 1 = 0 \) for a \( T_0 \leq a \), then

\[
(30) \quad x(t) + t + 1 < 0 \quad \text{for } t < T_0
\]

by (28) with \( b = T_0 \), and if \( x(T_1) + T_1 + 1 = 0 \) for a \( T_1 \geq a \), then

\[
(31) \quad x(t) + t + 1 < 0 \quad \text{for } t > T_1
\]

by (29) with \( c = T_1 \). Therefore the following five cases for \( x \) can occur:

**Case 1.** Let \( -t - 1 < x(t) < -t \) for \( t \in \mathbb{R} \). Then \( -x(t) - 1 < x(x(t)) < -x(t) \) for \( t \in \mathbb{R} \), and consequently

\[
-1 < x(x(t)) + x(t) = x'(t) < 0, \quad x''(t) = x'(t)(1 + x'(x(t))) < 0
\]

on \( \mathbb{R} \). Since \( 2a = x'(a) > -1 \) and \( x' \) is decreasing on \( \mathbb{R} \), \( x'(t) > 2a \) for \( t \in (-\infty, a) \). Then

\[
x(a) - x(t) = \int_t^a x'(s) \, ds > 2a(a - t)
\]

and \( x(t) < x(a) - 2a(a - t) \) for \( t < a \). Hence \( -t - 1 < x(a) - 2a(a - t) \) on \( (-\infty, a) \), which contradicts \( 2a > -1 \).

**Case 2.** Let \( x(t) < -t - 1 \) for \( t \in \mathbb{R} \setminus \{ a \} \). Then \( x'(t) = x(x(t)) + x(t) < -1 \) for \( t \in \mathbb{R} \setminus \{ a \} \) and so \( x''(t) = x'(t)(1 + x'(x(t))) > 0 \) for all \( t \neq a \). Let
Since \( x'(a) = 2a \leq -1 \) and \( x' \) is increasing, \( x'(t) < x'(t_0) < -1 \) for \( t < t_0 \) and therefore \( x(t) > x(t_0) + x'(t_0)(t - t_0) \) on \((-\infty, t_0)\). Then

\[-t - 1 > x(t_0) + x'(t_0)(t - t_0)\]

for \( t < t_0 \), which contradicts \( x'(t_0) < -1 \).

**Case 3.** There exists a \( T_0 \in (a, \infty) \) such that

\[-t - 1 < x(t) < -t \quad \text{for} \quad t < T_0 \quad \text{and} \quad x(t) < -t - 1 \quad \text{for} \quad t > T_0.\]

Let \( t_0 > T_0 \). By (29) (with \( c = t_0 \)),

\[(32) \quad x(t) + t + 1 \leq Ae^{t-t_0} \quad \text{for} \quad t \geq t_0,
\]

where \( A = x(t_0) + t_0 + 1 < 0 \). Let \( t_1 < a \) be a number such that \( x(t) \geq t_0 \) for \( t \leq t_1 \). Then (cf. (32))

\[(33) \quad x(x(t)) + x(t) + 1 \leq Ae^{x(t)-t_0} \quad \text{for} \quad t \leq t_1,
\]

and consequently

\[x'(t) = x(x(t) + x(t) \leq -1 + Ae^{x(t)-t_0} \quad \text{for} \quad t \leq t_1.
\]

Since

\[\lim_{t \to -\infty} (-1 + Ae^{x(t)-t_0}) = -\infty,
\]

we conclude that \( \lim_{t \to -\infty} x'(t) = -\infty \), which contradicts \(-t - 1 < x(t) < -t \) for \( t \in (-\infty, T_0) \).

**Case 4.** There exists a \( T_1 \in (-\infty, a) \) such that

\[x(t) < -t - 1 \quad \text{for} \quad t < T_1 \quad \text{and} \quad -t - 1 < x(t) < -t \quad \text{for} \quad t > T_1.
\]

Since \( x'(\tau) = -1 \) for a \( \tau \in \mathbb{R} \) if and only if \( x(\xi) = -\xi - 1 \) for \( \xi = x(\tau) \) and the equation \( x(t) = -t - 1 \) has the unique solution \( t = T_1 \), we deduce that \( x'(t_0) = -1 \) for the unique \( t_0 = x^{-1}(T_1) \in (a, \infty) \). Therefore \( x'(t) > -1 \) on \((-\infty, t_0)\) since \( x'(T_1) \geq -1 \), and then \( x''(t) = x'(t)(1 + x'(x(t))) < 0 \) for all \( t \in \mathbb{R} \) such that \( x(t) < t_0 \); that is, for all \( t > x^{-1}(t_0) \) \((< a)\). This proves that \( x' \) is decreasing on \((x^{-1}(t_0), \infty) \supset [t_0, \infty)\), and consequently \( x'(t) \leq x'(t_0 + 1) < -1 \) for \( t \in [t_0 + 1, \infty) \), which contradicts \( x(t) > -t - 1 \) for \( t > T_1 \).

**Case 5.** There exist \( T_2, T_3 \in \mathbb{R} \) such that \( T_2 < a < T_3 \) and

\[x(t) < -t - 1 \quad \text{for} \quad t \in (-\infty, T_2) \cup (T_3, \infty).
\]
Let $t_0 > T_3$. Then (32) holds with an $A < 0$ and using the same procedure as in Case 3, we obtain $\lim_{t \to -\infty} x'(t) = -\infty$, which contradicts $x(t) < -t - 1$ on $(-\infty, T_2)$.

Finally, we show that $a \in (-\frac{1}{2}, 0)$. Since $\left(x(x(t))\right)'_{t=a} \leq 1$, we have $2a \geq -1$. If $a = -\frac{1}{2}$, using the same procedure as in the end of the proof of Lemma 8 we obtain a contradiction. □

**Theorem 2.** Let $x \in \mathcal{A}_-$ and $x(a) = a$ for an $a \in \mathbb{R}$. Then $x \in \mathcal{A}_-^1$ and only if $a \in (-\frac{1}{2}, 0)$ and $x(t) > t$ for $t < a$, $x(t) < t$ for $t > a$.

The proof follows from Lemma 8 and Lemma 9.

**Definition 5.** Numbers $\alpha, \beta, \alpha \neq \beta$ are called conjugate points of an $x \in \mathcal{A}_-$ if $x(\alpha) = \beta$ and $x(\beta) = \alpha$. We call $\alpha, \beta \in \mathbb{R}$ conjugate points of (1) if $\alpha, \beta$ are conjugate points of an $x \in \mathcal{A}_-$.

**Corollary 5.** Let $x \in \mathcal{A}_-$. Then $x \in \mathcal{A}_-^2$ if and only if there exist conjugate points of $x$. If $\alpha < \beta$ are conjugate points of an $x \in \mathcal{A}_-^2$, then there exists a $D$-periodic continuation of $x$ with the period $\beta - \alpha$.

*Proof.* Let $\alpha < \beta$ be conjugate points of $x \in \mathcal{A}_-$. Then $x(\alpha)) = \alpha$ and $\beta = x(\alpha) \neq \alpha$, and consequently $x \in \mathcal{A}_-^2$ by Theorem 2.

Let $x \in \mathcal{A}_-^2$. By Theorem 2, there exists an $\alpha \in \mathbb{R}$ such that $x(\alpha) = \alpha$ and $x(\alpha) \neq \alpha$. Set $\beta = x(\alpha)$. Then $\beta \neq \alpha$ and $x(\beta) = \alpha$. So $\alpha, \beta$ are conjugate points of $x$.

Let $\alpha < \beta$ be conjugate points of an $x \in \mathcal{A}_-^2$. By Lemma 4, there exists a $D$-periodic continuation $y$ of $x$ with the period $\beta - \alpha$. □

**Theorem 3.** For any $0 < B \leq A$, there exists an $x \in \mathcal{A}_-^1$ such that $x(A) = -B$ and $x(t) < B$ for $t \in \mathbb{R}$.

*Proof.* Fix $0 < B \leq A$. Let $\mathbf{X}$ be the Fréchet space of $C^1$-functions on the interval $I = (-\infty, A]$ equipped with the topology of locally uniformly convergence of the functions and of their derivatives on $I$. Let

$$
\mathcal{D} = \left\{ x; \ x \in \mathbf{X}, \ x(A) = -B, \ x(t) \leq B, \ -2B \leq x'(t) \leq 0 \text{ for } t \in I \right\}.
$$

Then $\mathcal{D}$ is a bounded closed convex subset of $\mathbf{X}$ and since, for example, the function $B(1 - 2e^{k(t-A)})$ belongs to $\mathcal{D}$ for each $k \in (0, 1)$, $\mathcal{D}$ is a nonempty set. For each $x \in \mathcal{D}$, $x : I \to [-B, B] \subset I$. Define the operator $T : \mathcal{D} \to \mathbf{X}$ by

$$(Tx)(t) = -e^t\left(Be^{-A} + \int_{t}^{A} e^{-s}x(s)ds\right).$$

We see that any fixed point $x_0$ of $T$ is a solution of (1) on $I$ and $x_0(A) = -B$, $x_0'(t) \leq 0$, $x_0(t) \leq B$ for $t \in I$. By Theorem 1, there exists a right continuation $x$ of $x_0$ on $\mathbb{R}$ and, moreover, $x'(t) < 0$ on $\mathbb{R}$ by Lemma 2.
Then \( x(t) < B \) on \( \mathbb{R} \) and \( x \in \mathcal{A}^1 \). To prove the existence of a fixed point of \( T \) we use the Tikhonov–Schauder fixed point theorem. Let \( x \in \mathbb{X} \) and set \( y = T(x) \). Then \( y(A) = -B \) and since \( x'(t) \leq 0 \), \( x(t) \leq B \) on \( I \) and \( x(A) = -B \), there exists \( \lim_{t \to -\infty} x(t) \) and

\[
\lim_{t \to -\infty} x(t) \geq x(B) \geq x(A) = -B.
\]

By the L’Hospital rule,

\[
\lim_{t \to -\infty} y(t) = - \lim_{t \to -\infty} \frac{B e^{-A} + \int_t^A e^{-s} x(s) \, ds}{e^{-t}} = - \lim_{t \to -\infty} x(t)
\]

and (cf. (34))

\[
\lim_{t \to -\infty} y(t) \leq B.
\]

Using the equalities

\[
y'(t) = y(t) + x(x(t)), \quad y''(t) = y'(t) + (x(x(t)))', \quad t \in I,
\]

we obtain

\[
y'(t) = e^t \left( y'(A) e^{-A} - \int_t^A e^{-s} (x(x(s)))' \, ds \right), \quad t \in I,
\]

and \( y'(A) = y(A) + x(x(A)) = -B + x(-B) \leq 0 \). Thus, from the second equality of (36) and the inequality \((x(x(t)))' \geq 0\) on \( I \), we obtain \( y'(t) \leq 0 \) for \( t \in I \); hence (cf. (35)) \( |y(t)| \leq B \) on \( I \) and (cf. the first equality of (36)) \( y'(t) \geq -2B \) for \( t \in I \). This proves \( y \in \mathcal{D} \), and consequently \( T : \mathcal{D} \to \mathcal{D} \).

Let \( \{x_n\} \subset \mathcal{D} \) be a convergent sequence in \( \mathbb{X} \), \( \lim_{n \to \infty} x_n = x \). Then \( |x'_n(t)| \leq 2B \) for \( t \in I \), \( n \in \mathbb{N} \), and from the relations

\[
|x_n(x_n(t)) - x(t)| \leq |x_n(x_n(t)) - x_n(x(t))| + |x_n(x(t)) - x(x(t))|
\]

\[
= |x'_n(\xi_n)||x_n(t) - x(t)| + |x_n(x(t)) - x(x(t))|
\]

\[
\leq 2B|x_n(t) - x(t)| + |x_n(x(t)) - x(x(t))|
\]

\((\xi_n \text{ lies between } x_n(t) \text{ and } x(t))\) we deduce that \( \lim_{n \to \infty} x_n(x_n(t)) = x(x(t)) \) locally uniformly on \( I \). Then

\[
\lim_{n \to \infty} e^t \int_t^A e^{-s} x_n(x(s)) \, ds = e^t \int_t^A e^{-s} x(x(s)) \, ds
\]

in \( \mathbb{X} \), which shows that \( \lim_{n \to \infty} T(x_n) = T(x) \), and consequently \( T \) is a continuous operator.
Let \( B = \{ x; x \in \mathcal{D} \cap C^2(I), \ |x''(t)| \leq 2B(1 + 2B) \text{ for } t \in I \} \). By the second equality of (36), \( T(x) \in B \) for each \( x \in \mathcal{D} \) and since \( B \) is a relatively compact subset of \( \mathcal{X} \), \( T(\mathcal{D}) \) is relatively compact subset of \( \mathcal{X} \) as well. Hence \( T \) is a completely continuous operator and applying the Tikhonov–Schauder fixed point theorem to \( T \), there exists a fixed point of \( T \).

**Remark 6.** Let \( A \in (0, \frac{1}{2}] \) and let \( x_0 \in \mathcal{A}_1 \) be such that \( x_0(A) = -A \). The existence of \( x_0 \in \mathcal{A}_1 \) follows from Theorem 3 with \( B = A \). Then \( x_0(t) \geq -A, x_0(x_0(t)) > -A \) for \( t \in (-\infty, A] \), and consequently \( x_0'(A) = x_0(A) + x_0(x_0(A)) > -2A \geq -1 \).

**Lemma 10.** Let \( x \in \mathcal{A}_1 \) and \( A = \lim_{t \to -\infty} x(t) \). Then
\[
 x(t) < -t, \quad x'(t) > -1 \quad \text{for } t \in (-\infty, A)
\]
and
\[
 x''(t) < 0, \quad x'''(t) < 0 \quad \text{for } t \in \mathbb{R}.
\]

**Proof.** By Lemma 6 and Lemma 8, \( x(A) = -A \) and there exists an \( a \in (-\frac{1}{2}, 0) \) such that \( x(a) = a \). Assume \( x'(t_0) = -1 \) for a \( t_0 \in (-\infty, A) \). Setting \( \xi = x(t_0) \), then \( \xi \in (-A, A) \) and \( x(\xi) = -\xi - 1 \). If \( \xi > a \), then \( x(t) < -t - 1 \) for \( t > \xi \) by (29) with \( c = \xi \); hence \( -A = x(A) < -A - 1 \), a contradiction. Let \( \xi \leq a \). By (28) (with \( b = \xi \)), \( x(t) < -t - 1 \) for \( t < \xi \), and so \( x'(t) = x(x(t)) + x(t) < -1 \) for all \( t \in \mathbb{R} \) such that \( x(t) < \xi \), that is, for all \( t > t_0 \). If \( x(t_0) \leq -t_0 \), then \( x(t) < -t \) for \( t > t_0 \), which contradicts \( x(A) = -A \). So \( x(t_0) > -t_0 \) and then the inequality \( x(t_0) < A \) implies \( t_0 \in (-A, A) \). Thus \( x(T) = t_0 \) for a (unique) \( T \in (-\infty, A) \) and therefore \( 0 > x(T) + x(x(T)) = t_0 + x(t_0) \), a contradiction. Hence \( x'(t) > -1 \) for \( t \in (-\infty, A) \) and then \( x(t) < -t \) on this interval since \( x(A) = -A \). Moreover, \( x'(x(t)) > -1 \) for \( t \in \mathbb{R} \) since \( x(t) \in (-\infty, A) \) for \( t \in \mathbb{R} \). Therefore \( x''(t) = x'(t)(1 + x'(x(t))) > 0 \) on \( \mathbb{R} \), and consequently \( x'''(t) = x''(t)(1 + x'(x(t))) + x'''(x(t))(x'(t))^2 < 0 \) on this interval.

**Lemma 11.** Let \( x \in \mathcal{A}_1 \) and \( x(a) = a \) for an \( a \in \mathbb{R} \). If \( y \in \mathcal{A}_1 \) and \( y(a) = a \), then \( x(t) = y(t) \) for \( t \in \mathbb{R} \).

**Proof.** By Lemma 8, \( a \in (-\frac{1}{2}, 0) \). Assume, on the contrary, that there exists a solution \( y \in \mathcal{A}_1 \) such that \( y(a) = a \) and \( x \neq y \). By Lemma 3, there exists an \( \varepsilon > 0 \) such that \( x(t) = y(t) \) for \( t \in [a - \varepsilon, a + \varepsilon] \). Assume \( x(t) = y(t) \) for \( t \in (-\infty, a + \varepsilon) \) and set \( \beta = \inf \{ t; t \geq a + \varepsilon, \ x(t) \neq y(t) \} \). Then \( \infty \geq \beta \geq a + \varepsilon \) and
\[
\max \{ |x(t) - y(t)|; \beta \leq t \leq \beta + \rho \} > 0 \quad \text{for } \rho > 0.
\]
Consider the initial Cauchy problems

\begin{align}
\frac{dz}{dt} &= x(z) + z, \quad z(\beta) = x(\beta) \\
\frac{dw}{dt} &= y(w) + w, \quad w(\beta) = y(\beta).
\end{align}

Since \( x, y \in C^1(\mathbb{R}) \), there exist the unique solutions \( z_1 \) and \( w_1 \) of (38) and (39) on a neighbourhood \( U \) of the point \( t = \beta \), respectively. We know that \( x(\beta) = y(\beta) \in (-\infty, a) \subset (-\infty, \beta] \) and \( x \) (resp. \( y \)) is a solution of (38) (resp. (39)) on \( \mathbb{R} \). By the uniqueness theorem for ordinary differential equations, \( z_1(t) = x(t), w_1(t) = y(t) \) for \( t \in U \) and then \( z_1(t) = w_1(t) \) for \( t \in I = \{t; t \in U, z_1(t) \in (-\infty, \beta]\} \); hence \( x(t) = y(t) \) for \( t \in (-\infty, \beta] \cup I \supset (-\infty, \beta_1] \) with a \( \beta_1 > \beta \), which contradicts (37). Similarly for \( x(t) = y(t) \) on \([a - \varepsilon, \infty)\). We have proved that

\[
T_1 = \inf \{t; t \leq a - \varepsilon, x(s) = y(s) \text{ for } s \in [t, a]\} > -\infty
\]

and

\[
T_2 = \sup \{t; t \geq a + \varepsilon, x(s) = y(s) \text{ for } s \in [a, t]\} < \infty.
\]

Clearly,

\begin{align}
[a - \varepsilon, a + \varepsilon] &\subset [T_1, T_2] \\
\frac{dx}{dt} &= y(t) \quad \text{for } t \in [T_1, T_2].
\end{align}

We now prove that \( x(T_1) = T_2 \) and \( x(T_2) = T_1 \). Evidently \( x(x(t)) = y(y(t)) \) on \([T_1, T_2] \), which gives (cf. (41) and the definitions of \( T_1, T_2 \))

\[
x, y : [T_1, T_2] \rightarrow [T_1, T_2].
\]

Assume \( x(T_1) < T_2 \) (the case where \( x(T_2) > T_1 \) treats similarly) and consider the initial Cauchy problems

\begin{align}
\frac{dz}{dt} &= x(z) + z, \quad z(T_1) = x(T_1) \\
\frac{dw}{dt} &= y(w) + w, \quad w(T_1) = y(T_1).
\end{align}
Since \( x, y \in C^1(\mathbb{R}) \) and \( x(T_1) = y(T_1) \in (T_1, T_2) \), there exists a unique solution \( z_2 \) (resp. \( w_2 \)) of (42) (resp. (43)) defined on a neighbourhood \( \mathcal{V} = (T_1 - \nu, T_1 + \nu) \) of the point \( T_1 \), where \( \nu \) is a positive number. We know that \( x \) and \( y \) are solutions of (42) and (43) on \( \mathbb{R} \), respectively. Thus \( x(t) = z_2(t) \), \( y(t) = w_2(t) \) for \( t \in \mathcal{V} \) and (cf. (41)) \( z_2(t) = w_2(t) \) on \( J = \{ t; t \in \mathcal{V}, z_2(t) \in [T_1, T_2] \} \). It follows that \( x(t) = y(t) \) for \( t \in [T_1, T_2] \cup J \) and since \( [T_1, T_2] \cup J \neq [T_1, T_2] \) we obtain a contradiction to the definition of the numbers \( T_1 \) and \( T_2 \).

\begin{theorem}
Let \( x \in \mathcal{A}^1_-, A = \lim_{t \to -\infty} x(t) \) and \( x(a) = a \) for an \( a \in \mathbb{R} \). Then
\[
A < \frac{4}{5} \quad \text{and} \quad -\frac{1}{4} < a < 0.
\]
\end{theorem}

\begin{proof}
By Lemma 6 and Lemma 10, \( x(A) = -A \), \( x''(t) < 0 \) and \( x'''(t) < 0 \) on \( \mathbb{R} \), and consequently \( -A < a \), \( x'(t) > x'(a) = 2a \) for \( t \in (-\infty, a) \), \( x'(t) < 2a \), \( x''(t) < x''(a) = 2a(1 + 2a) \) for \( t > a \). Thus
\[
x'(t) = x'(a) + \int_a^t x''(s) \, ds < 2a + 2a(1 + 2a)(t - a)
\]
and
\[
(44) \quad x(t) = x(a) + \int_a^t x'(s) \, ds < a + 2a(t - a) + a(1 + 2a)(t - a)^2
\]
for \( t > a \). Then (44) (with \( t = A \)) implies
\[
-A < a + 2a(A - a) + a(1 + 2a)(A - a)^2,
\]
and therefore \( T = A - a \) (\( > 0 \)) satisfied the quadratic inequality
\[
-a(1 + 2a)T^2 - (1 + 2a)T - 2a < 0.
\]
Consider the quadratic equation \( p(t) = -a(1 + 2a)t^2 - (1 + 2a)t - 2a = 0 \).
By Lemma 8, \( 1 + 2a > 0 \), \( -a(1 + 2a) > 0 \), and therefore \( \lim_{t \to -\infty} p(t) = \infty \).
Then discriminant \( D = (1 + 2a)(1 + 2a - 8a^2) \) of the quadratic equation \( p(t) = 0 \) is necessarily positive, which implies \( a \in (-\frac{1}{4}, 0) \).

We next have
\[
x(a) = x(t) + \int_t^a x'(s) \, ds > x(t) + 2a(a - t)
\]
and thus \( x(t) < a - 2a(a - t) \) for \( t \in (-\infty, a) \); in particular \( x(-A) < a - 2a(a + A) \). Since \( x'(A) \geq -1 \) (see Lemma 10), \( -1 \leq x'(A) = x(x(A)) + x(A) = \)

Global Properties of Decreasing Solutions

Let $x(-A)-A$ and therefore $x(-A) \geq A-1$. Then $A-1 \leq x(-A) < a-2a(a+A)$ and $A < \frac{1+a-2a^2}{1+2a} = 1-a$; hence $A < \frac{5}{4}$.

Denote by $I_i^-$ $(i = 1, 2)$ the set of all $a \in (-\infty, 0)$ such that $x(a) = a$ for an $x \in A_i^-$. 

Remark 7. By Theorem 4, $I_i^+ \subset \left( -\frac{1}{4}, 0 \right)$.

Lemma 12. $I_i^- \cup \{0\}$ is a closed subset of $\mathbb{R}$.

Proof. Let $\{a_n\} \subset I_i^-$ be a convergent sequence, $\lim_{n \to \infty} a_n = a > 0$. We have to show that $a \in I_i^-$. By the definition of $I_i^-$, there exists a sequence $\{x_n\} \subset A_i^-$ such that $x_n(a_n) = a_n$ for $n \in \mathbb{N}$. Let $A_n = \lim_{t \to -\infty} x_n(t)$. By Lemmas 4, 6 and 8 and Theorem 4, $\{a_n\} \subset \left( -\frac{1}{4}, 0 \right)$, $\{A_n\} \subset (0, \frac{5}{4})$, $x_n(A_n) = -A_n$, $-A_n < x_n(t) < A_n$, $-1 < x_n(t) < 0$ for $t \in (-\infty, A_n)$ and, moreover, $x_n(x_n(t)) \geq a_n$ for $t \in [a_n, \infty)$. Then (cf. (3) with $c = A_n$)

$$x_n(t) = e^t(-A_n e^{-A_n} + \int_{A_n}^t e^{-s} x_n(x_n(s)) \, ds) \geq e^t(-A_n e^{-A_n} + a_n \int_{A_n}^t e^{-s} \, ds)$$

for $t \in [A_n, \infty)$, $n \in \mathbb{N}$; hence

$$x_n(2 + A_n) \geq e^{2 + A_n}(-A_n e^{-A_n} + a_n e^{-A_n} - a_n a^{-2 - A_n})$$

$$= (a_n - A_n)e^2 - a_n > (-\frac{1}{4} - \frac{5}{4})e^2 = -\frac{3}{2}e^2.$$ 

Then

$$\frac{5}{4} > x_n(t) \geq -\frac{3}{2}e^2 \quad \text{for} \quad t \in (-\infty, 2 + A_n], \quad n \in \mathbb{N},$$

and since $-\frac{5}{4} < x_n(x_n(t)) < \frac{5}{4}$ for $t \in \mathbb{R}$ and $n \in \mathbb{N}$ we have $0 > x_n'(t) = x_n(x_n(t)) + x_n(t) > -\frac{5}{4} - \frac{3}{2}e^2$ for $t \in (-\infty, 2 + A_n]$, and $n \in \mathbb{N}$. Set $B = \sup\{A_n; \, n \in \mathbb{N}\}$. Then $B + \frac{3}{4} \leq 2 + A_n$ for all $n \in \mathbb{N}$ and $\frac{5}{4} > x_n(t) \geq -\frac{3}{2}e^2$, $-\frac{5}{4} < x_n(x_n(t)) < \frac{5}{4}$, $0 > x_n'(t) > -\frac{5}{4} - \frac{3}{2}e^2$ for $t \in (-\infty, B + \frac{3}{4})$, $n \in \mathbb{N}$. By the Arzelà–Ascoli theorem, we can assume that $\{x_n(t)\}$ locally uniformly convergent on $(-\infty, B + \frac{3}{4})$, $\lim_{n \to \infty} x_n = x(t)$. Evidently, $x$ maps $(-\infty, B + \frac{3}{4})$ into $(-\infty, B]$. Next, we have (for $t \in (-\infty, B + \frac{3}{4})$)

$$|x_n(x_n(t)) - x(x(t))| \leq |x_n(x_n(t)) - x_n(x(t))| + |x_n(x(t)) - x(x(t))|$$

$$= |x_n'(\xi)||x_n(t) - x(t)| + |x_n(x(t)) - x(x(t))| \leq \left( \frac{5}{4} + \frac{3}{2}e^2 \right)|x_n(t) - x(t)|$$

$$+ |x_n(x(t)) - x(x(t))|,$$

where $\xi$ lies between $x_n(t)$ and $x(t)$. So $\lim_{n \to \infty} x_n(x_n(t)) = x(x(t))$ locally uniformly on $(-\infty, B + \frac{3}{4})$. Taking the limit in the equalities

$$x_n(t) = e^t(a_n e^{-a_n} + \int_{a_n}^t e^{-s} x_n(x_n(s)) \, ds), \quad t \in (-\infty, B + \frac{3}{4})$$

and therefore $x(-A) \geq A-1$. Then $A-1 \leq x(-A) < a-2a(a+A)$ and $A < \frac{1+a-2a^2}{1+2a} = 1-a$; hence $A < \frac{5}{4}$. 

Remark 7. By Theorem 4, $I_i^+ \subset \left( -\frac{1}{4}, 0 \right)$.

Lemma 12. $I_i^- \cup \{0\}$ is a closed subset of $\mathbb{R}$.

Proof. Let $\{a_n\} \subset I_i^-$ be a convergent sequence, $\lim_{n \to \infty} a_n = a > 0$. We have to show that $a \in I_i^-$. By the definition of $I_i^-$, there exists a sequence $\{x_n\} \subset A_i^-$ such that $x_n(a_n) = a_n$ for $n \in \mathbb{N}$. Let $A_n = \lim_{t \to -\infty} x_n(t)$. By Lemmas 4, 6 and 8 and Theorem 4, $\{a_n\} \subset \left( -\frac{1}{4}, 0 \right)$, $\{A_n\} \subset (0, \frac{5}{4})$, $x_n(A_n) = -A_n$, $-A_n < x_n(t) < A_n$, $-1 < x_n'(t) < 0$ for $t \in (-\infty, A_n)$ and, moreover, $x_n(x_n(t)) \geq a_n$ for $t \in [a_n, \infty)$. Then (cf. (3) with $c = A_n$)

$$x_n(t) = e^t(-A_n e^{-A_n} + \int_{A_n}^t e^{-s} x_n(x_n(s)) \, ds) \geq e^t(-A_n e^{-A_n} + a_n \int_{A_n}^t e^{-s} \, ds)$$

for $t \in [A_n, \infty)$, $n \in \mathbb{N}$; hence

$$x_n(2 + A_n) \geq e^{2 + A_n}(-A_n e^{-A_n} + a_n e^{-A_n} - a_n a^{-2 - A_n})$$

$$= (a_n - A_n)e^2 - a_n > (-\frac{1}{4} - \frac{5}{4})e^2 = -\frac{3}{2}e^2.$$ 

Then

$$\frac{5}{4} > x_n(t) \geq -\frac{3}{2}e^2 \quad \text{for} \quad t \in (-\infty, 2 + A_n], \quad n \in \mathbb{N},$$

and since $-\frac{5}{4} < x_n(x_n(t)) < \frac{5}{4}$ for $t \in \mathbb{R}$ and $n \in \mathbb{N}$ we have $0 > x_n'(t) = x_n(x_n(t)) + x_n(t) > -\frac{5}{4} - \frac{3}{2}e^2$ for $t \in (-\infty, 2 + A_n]$, and $n \in \mathbb{N}$. Set $B = \sup\{A_n; \, n \in \mathbb{N}\}$. Then $B + \frac{3}{4} \leq 2 + A_n$ for all $n \in \mathbb{N}$ and $\frac{5}{4} > x_n(t) \geq -\frac{3}{2}e^2$, $-\frac{5}{4} < x_n(x_n(t)) < \frac{5}{4}$, $0 > x_n'(t) > -\frac{5}{4} - \frac{3}{2}e^2$ for $t \in (-\infty, B + \frac{3}{4})$, $n \in \mathbb{N}$. By the Arzelà–Ascoli theorem, we can assume that $\{x_n(t)\}$ locally uniformly convergent on $(-\infty, B + \frac{3}{4})$, $\lim_{n \to \infty} x_n = x(t)$. Evidently, $x$ maps $(-\infty, B + \frac{3}{4})$ into $(-\infty, B]$. Next, we have (for $t \in (-\infty, B + \frac{3}{4})$)

$$|x_n(x_n(t)) - x(x(t))| \leq |x_n(x_n(t)) - x_n(x(t))| + |x_n(x(t)) - x(x(t))|$$

$$= |x_n'(\xi)||x_n(t) - x(t)| + |x_n(x(t)) - x(x(t))| \leq \left( \frac{5}{4} + \frac{3}{2}e^2 \right)|x_n(t) - x(t)|$$

$$+ |x_n(x(t)) - x(x(t))|,$$

where $\xi$ lies between $x_n(t)$ and $x(t)$. So $\lim_{n \to \infty} x_n(x_n(t)) = x(x(t))$ locally uniformly on $(-\infty, B + \frac{3}{4})$. Taking the limit in the equalities

$$x_n(t) = e^t(a_n e^{-a_n} + \int_{a_n}^t e^{-s} x_n(x_n(s)) \, ds), \quad t \in (-\infty, B + \frac{3}{4})$$
as \( n \to \infty \) we obtain

\[
x(t) = e^t (ae^{-a} + \int_a^t e^{-s} x(s) \, ds), \quad t \in (-\infty, B + \frac{3}{4}];
\]

hence \( x(a) = a, x'(t) = x(x(t)) + x(t) \) for \( t \in (-\infty, B + \frac{3}{4}] \). Then there exists an \( x_0 \in \mathcal{A}_- \) such that \( x_0(t) = x(t) \) on \( (-\infty, B + \frac{3}{4}] \) and therefore \( a \in I^- \). \( \Box \)

**Theorem 5.** (A summary). Any maximal decreasing solution \( x \) of (1) is defined on \( \mathbb{R} \) and \( \lim_{t \to -\infty} x(t) = -\infty \). The set \( \mathcal{A}_- \) of all maximal decreasing solutions of (1) is the union of nonempty sets

\[
\mathcal{A}_1^- = \{ x; x \in \mathcal{A}_-, \lim_{t \to -\infty} x(t) < \infty \}
\]

and

\[
\mathcal{A}_2^- = \{ x; x \in \mathcal{A}_-, \lim_{t \to -\infty} x(t) = \infty \}.
\]

For each \( a \in [-\frac{1}{2}, 0) \), there exists an \( x \in \mathcal{A}_- \) such that \( x(a) = a \). If \( x \in \mathcal{A}_- \), then

(a) \( x \in \mathcal{A}_1^- \) if and only if \( x(x(t)) > t \) for \( t < a \) and \( x(x(t)) < t \) for \( t > a \), where \( a \in (-\frac{1}{2}, 0) \) and \( x(a) = a \),

(b) \( x \in \mathcal{A}_2^- \) if and only if \( x(t) < -t \) for \( t \in \mathbb{R} \),

(c) \( x \in \mathcal{A}_2^- \) if and only if there exist conjugate points of \( x \),

(d) \( x \in \mathcal{A}_2^- \) if and only if there exists a D-periodic continuation of \( x \).

If \( x \in \mathcal{A}_1^- \) and \( A = \lim_{t \to -\infty} x(t) \), \( x(a) = a \), then

(i) \( x(A) = -A, x(t) < -t, x'(t) > -1 \) for \( t \in (-\infty, A) \),

(ii) \( x''(t) < 0, x'''(t) < 0 \) for \( t \in \mathbb{R} \),

(iii) \( A \in (0, \frac{5}{4}), a \in (-\frac{1}{4}, 0) \).

Furthermore,

(j) \( I_1^- \cup \{0\} \) is a compact set, where

\[
I_1^- = \{ a; a \in \mathbb{R}, x(a) = a \text{ for an } x \in \mathcal{A}_1^- \},
\]

(jj) for any \( a \in I_1^- \), there exists a unique \( x \in \mathcal{A}_- \) such that \( x(a) = a \),

(jj) for any \([A, -B] \in \mathbb{R}^2, 0 < B \leq A\), there exists an \( x \in \mathcal{A}_1^- \) such that \( x(A) = -B \).

4. **Open problems.** The structure of (maximal) decreasing solutions of (1) described above is not complete. Some open problems are stated below.

**Problem 1.** Is there a solution \( x \in \mathcal{A}_- \) satisfying \( x(a) = a \) for an/each \( a \in (-\infty, -\frac{1}{2}) \)?
We know (cf. Lemma 3 and Remark 1) that for each \( a \in [-\frac{1}{2}, 0) \) there exists an \( x \in \mathcal{A}_- \) such that \( x(a) = a \). For \( a \in (-\frac{1}{2}, 0) \) the existence result is proved by the Banach fixed point theorem (see [2]) and by Theorem 1. Observe that the existence of decreasing solutions of (W): \( x'(t) = f(x(x(t))) \) was proved for each \( a \in (-\infty, 0) \) in [3].

**Problem 2.** Is \( I_-^1 \) an interval?
We only know that \( I_-^1 \cup \{0\} \) is a compact set and \( I_-^1 \subset (-\frac{1}{4}, 0) \) by Remark 7 and Lemma 12.

**Problem 3.** For each \( a \in I_-^1 \), denote (a unique) \( x_a \in \mathcal{A}_-^1 \) such that \( x_a(a) = a \). Set \( \alpha(a) = \lim_{t \to -\infty} x_a(t) \) for \( a \in I_-^1 \). Is the set \( \{ \alpha(a); a \in I_-^1 \} \) an interval? Is \( \alpha \) a continuous and/or monotone function?

By Lemma 6, \( x_a(\alpha(a)) = -\alpha(a) \) for \( a \in I_-^1 \). If the set \( \mathcal{S} = \{ \alpha(a); a \in I_-^1 \} \) is an interval, then \( \mathcal{S} = (0, C] \) with an \( C, \frac{5}{4} > C > 0 \), and consequently for each \( 0 < C_1 \leq C \) there exists an \( x \in \mathcal{A}_-^1 \) such that \( \lim_{t \to -\infty} x(t) = C_1 \). If \( \alpha \) is a monotone (necessarily decreasing) function, then for any \( C_2 \in \mathcal{S} \) there exists a unique \( x \in \mathcal{A}_-^1 \) satisfying \( \lim_{t \to -\infty} x(t) = C_2 \).

**Problem 4.** Let \( A \in (0, \infty) \). Is there a unique \( x \in \mathcal{A}_-^1 \) such that \( x(A) = -A \) ?
We only know that for each \( A \in (0, \infty) \) there exists at least one solution \( x \in \mathcal{A}_-^1 \) satisfying \( x(A) = -A \) by Theorem 3 (with \( B = A \)).

**Problem 5.** Is any element of \( \mathcal{A}_-^2 \) a \( D \)-periodic solution of (1)?
We only know that for each \( x \in \mathcal{A}_-^2 \) there exist conjugate points of \( x \) and if \( \alpha < \beta \) are conjugate points of an \( x \in \mathcal{A}_-^2 \), then there exist a \( D \)-periodic continuation \( y \in \mathcal{A}_-^2 \) of \( x \) with the period \( \beta - \alpha \) such that \( x(t) = y(t) \) for \( t \in [\alpha, \beta] \) (see Lemma 4). If we prove that \( x(t) = y(t) \) for \( t \in \mathbb{R} \), then it is clear that any element of \( \mathcal{A}_-^2 \) is a \( D \)-periodic solution of (1).

**References**


