VARIATION FORMULAS OF SOLUTION FOR A NEUTRAL FUNCTIONAL-DIFFERENTIAL EQUATION TAKING INTO ACCOUNT DELAY FUNCTION PERTURBATION AND THE DISCONTINUOUS INITIAL CONDITION

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Abstract. Variation formulas of solution are obtained for linear with respect to pre-history of the phase velocity neutral functional-differential equations with variable delays and with the discontinuous initial condition. In the variation formulas are detected the effects of perturbation of delay function entering in the phase coordinates and the discontinuous initial condition. The variation formula of solution plays the basic role in proving of the necessary conditions of optimality and under sensitivity analysis of mathematical models. Discontinuity of the initial condition means that the values of the initial function and the trajectory, in general, do not coincide at the initial moment.

Key Words. Neutral functional-differential equation, variation formula of solution, effect of delay function perturbation, effect of the discontinuous initial condition.

AMS(MOS) subject classification. 34K38, 34K40, 34K27.

1. Introduction. For the neutral functional-differential equation

\[ \dot{x}(t) = A(t)\dot{x}(\sigma(t)) + f_0(t, x(t), x(\tau_0(t))) \]

with the discontinuous initial condition

\[ x(t) = \varphi_0(t), t < t_{00}, x(t_{00}) = x_{00} \]
linear representations of the main part of a solution increment (variation formulas) are obtained with respect to perturbations of initial moment $t_0$, initial function $\varphi_0(t)$, initial vector $x_{00}$, delay function $\tau_0(t)$ and nonlinear term $f_0$ of the right-hand side of equation (1.1). In the variation formulas are detected the effects of perturbation of delay function and discontinuous initial condition. The variation formula of solution plays the basic role in proving of the necessary conditions of optimality [1-3] and under sensitivity analysis of mathematical models [4,5]. Variation formulas of solution for various classes of neutral functional-differential equations without perturbation of delay are given in [6-8].

2. Variation formulas of solution. Let $I = [a, b]$ be a finite interval and $\mathbb{R}^n$ be the $n$-dimensional vector space of points $x = (x^1, ..., x^n)^T$, where $T$ is the sign of transposition. Suppose that $O \subset \mathbb{R}^n$ and $X_0 \subset O$ are open sets, and $E_f$ is the set of functions $f : I \times O^2 \to \mathbb{R}^n$ satisfying the following conditions: the function $f(t, \cdot) : O^2 \to \mathbb{R}^n$ is continuously differentiable for almost all $t \in I$; the functions $f(t, x, y), f_x(t, x, y)$ and $f_y(t, x, y)$ are measurable on $I$ for any $(x, y) \in O^2$; for each $f \in E_f$ and compact set $K \subset O$, there exists a function $m_{f,K}(t) \in L(I, \mathbb{R}^+) = [0, \infty)$, such that

$$|f(t, x, y)| + |f_x(t, x, y)| + |f_y(t, x, y)| \leq m_{f,K}(t)$$

for all $(x, y) \in K^2$ and almost all $t \in I$.

Further, let $D$ be the set of continuously differentiable scalar functions (delay functions) $\tau(t), t \in [a, \infty)$, satisfying the conditions:

$$\tau(t) \leq t, \quad e_2 \geq \dot{\tau}(t) \geq e_1 > 0, \quad \inf\{\tau(a) : \tau \in D\} := \hat{\tau} > -\infty,$$

$$\sup\{\tau^{-1}(b) : \tau \in D\} < +\infty,$$

where $e_1, e_2$ are given numbers and $\tau^{-1}(t)$ is the inverse function of $\tau(t)$. Let $\Phi$ be the set of continuously differentiable initial functions $\varphi(t) \in X_1, t \in I_1 = [\hat{\tau}, b]$, where $X_1 \subset O$ is an open set.

To each element

$$\mu = (t_0, x_0, \tau, \varphi, f) \in \Lambda = [a, b] \times X_0 \times D \times \Phi \times E_f$$

we assign the neutral functional-differential equation

$$\dot{x}(t) = A(t)\dot{x}(\sigma(t)) + f(t, x(t), x(\tau(t)))$$

with the initial condition

$$x(t) = \varphi(t), t \in [\hat{\tau}, t_0), x(t_0) = x_0.$$
where $A(t)$ is a given continuous matrix function with dimension $n \times n$; $\sigma \in D$ is a fixed delay function. The condition (2.2) is said to be the discontinuous initial condition since, in general, $\varphi(t_0) \neq x_0$.

**DEFINITION 2.1.** Let $\mu = (t_0, x_0, \tau, \varphi, f) \in \Lambda$. A function $x(t) = x(t; \mu) \in O$, $t \in [\hat{\tau}, t_1]$, $t_1 \in (t_0, b]$ is called the solution of equation (2.1) with the initial condition (2.2) or solution corresponding to the element $\mu$, if it satisfies the initial condition (2.2) and on the interval $[t_0, t_1]$ is absolutely continuous and satisfies equation (2.1) almost everywhere.

Let $\mu_0 = (t_{00}, x_{00}, \tau_0, \varphi_0, f_0) \in \Lambda$ be a given element and $x_0(t)$ be the solution corresponding to $\mu_0$ and defined on $[\hat{\tau}, t_{10}]$, with $a < t_{00} < t_{10} < b$.

Let us introduce the set of variations

$$V = \left\{ \delta \mu = (\delta t_0, \delta x_0, \delta \tau, \delta \varphi, \delta f) : | \delta t_0 | \leq \alpha, | \delta x_0 | \leq \alpha, \| \delta \tau \| \leq \alpha, \right\}$$

$$\delta \varphi = \sum_{i=1}^{k} \lambda_i \delta \varphi_i, \delta f = \sum_{i=1}^{k} \lambda_i \delta f_i, | \lambda_i | \leq \alpha, i = 1, k$$

Here

$$\delta t_0 \in I - t_{00}, \delta x_0 \in X_0 - x_{00}, \delta \tau \in D - \tau_0, \| \delta \tau \| = \sup \{|\delta \tau(t)| : t \in I\}$$

and

$$\delta \varphi_i \in \Phi - \varphi_0, \delta f_i \in E_f - f_0, i = 1, k$$

are fixed functions, $\alpha > 0$ is a fixed number.

There exist numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for arbitrary $(\varepsilon, \delta \mu) \in (0, \varepsilon_1] \times V$ the element $\mu_0 + \varepsilon \delta \mu \in \Lambda$ and there corresponds the solution $x(t; \mu_0 + \varepsilon \delta \mu)$ defined on the interval $[\hat{\tau}, t_{10} + \delta_1] \subset I_1$ ( [9], Theorem 2).

Due to the uniqueness, the solution $x(t; \mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\hat{\tau}, t_{10} + \delta_1]$. Therefore, the solution $x_0(t)$ is assumed to be defined on the interval $[\hat{\tau}, t_{10} + \delta_1]$.

Let us define the increment of the solution $x_0(t) = x(t; \mu_0)$:

$$\Delta x(t; \varepsilon \delta \mu) = x(t; \mu_0 + \varepsilon \delta \mu) - x_0(t),$$

$$\forall (t, \varepsilon, \delta \mu) \in [\hat{\tau}, t_{10} + \delta_1] \times (0, \varepsilon_1] \times V.$$

**THEOREM 2.1.** Let the following conditions hold:

1. $\gamma_0(t_{00}) < t_{10}$, where $\gamma_0(t)$ is the inverse function of $\tau_0(t)$;
2. For each compact set $K \subset O$ there exists a number $m_K > 0$ such that

$$|f_0(z)| \leq m_K, \forall z = (t, x, y) \in I \times K^2;$$
2.3. There exist the limits

\[ \lim_{z \to z_0} f_0(z) = f_0^-, \quad z \in (a, t_{00}] \times O^2, \]

\[ \lim_{(z_1, z_2) \to (z_{10}, z_{20})} [f_0(z_1) - f_0(z_2)] = f_{01}^-, \quad z_i \in (t_{00}, \gamma_0(t_{00})) \times O^2, \quad i = 1, 2, \]

where

\[ z_0 = (t_{00}, x_{00}, \varphi_0(\tau_0(t_{00}))), \quad z_{10} = (\gamma_0(t_{00}), x_0(\gamma_0(t_{00})), x_{00}), \]

\[ z_{20} = (\gamma_0(t_{00}), x_0(\gamma_0(t_{00})), \varphi_0(t_{00})). \]

Then there exist numbers \( \varepsilon_2 \in (0, \varepsilon_1) \) and \( \delta_2 \in (0, \delta_1) \) such that for arbitrary

\( (t, \varepsilon, \delta \mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times V^- \),

where \( V^- = \{ \delta \mu \in V : \delta t_0 \leq 0, \delta \tau(t_{00}) > 0 \} \) we have

\[ (2.3) \]

\[ \Delta x(t; \varepsilon \delta \mu) = \varepsilon \delta x(t; \delta \mu) + o(t; \varepsilon \delta \mu), \]

where

\[ \delta x(t; \delta \mu) = \{ Y(t_{00}--; t) [\dot{\varphi}_0(t_{00}) - A(t_{00}) \dot{\varphi}_0(\sigma(t_{00}))] - f_0^- \} \]

\[ Y(\gamma_0(t_{00})--; t f_{01^-} \gamma_0(t_{00})) \} \delta t_0 + Y(\gamma_0(t_{00})--; t f_{01^-} \gamma_0(t_{00}) \delta \tau(\gamma_0(t_{00}))) + \]

\( (2.4) \)

\[ \beta(t; \delta \mu), \]

and

\[ \beta(t; \delta \mu) = \Psi(t_{00}; t) \left[ \delta x_0 - \dot{\varphi}_0(t_{00}) \delta t_0 \right] + \int_{t_{00}}^t Y(s; t f_{0y}[s] x_0(\tau(t(s))) \delta \tau(s) ds + \int_{t_{00}}^{t_{00}} Y(\gamma_0(s); t f_{0y}[\gamma_0(s)] \gamma_0(s) \delta \varphi(s) ds + \int_{\sigma(t_{00})}^{t_{00}} Y(\nu(s); t) A(\nu(s)) \dot{\nu}(s) \delta \varphi(s) ds + \]

\( (2.5) \)

\[ \int_{t_{00}}^t Y(s; t) \delta f[s] ds. \]

Here, \( \Psi(s; t) \) and \( Y(s; t) \) are \( n \times n \) matrix functions satisfying the system

\[ \left\{ \begin{array}{l} \Psi_s(s; t) = -Y(s; t f_{0x}[t] - Y(\gamma_0(s); t f_{0y}[\gamma_0(s)] \gamma_0(s), \\
Y(s; t) = \Psi(s; t) + Y(\nu(s); t) A(\nu(s)) \nu(s)), \quad s \in [t_{00} - \delta_2, t_{10}] \end{array} \right. \]
and the condition
\[ \Psi(s; t) = Y(s; t) = \begin{cases} H, & s = t \\ \Theta, & s > t; \end{cases} \]

is the identity matrix and \( \Theta \) is the zero matrix, \( \nu(s) \) is the inverse function of \( \sigma(s) \),

\[ f_{0x}[s] = f_{0x}(s, x_0(s), x_0(\tau_0(s))), \delta f[s] = \delta f(s, x_0(s), x_0(\tau_0(s))). \]

SOME COMMENTS. The function \( \delta x(t; \delta \mu) \) is called the variation of the solution \( x_0(t), t \in [t_{10} - \delta_2, t_{10} + \delta_2] \), and the expression (2.4) is called the variation formula.

Theorem 2.1 corresponds to the case when the variation at the point \( t_{00} \) is performed on the left.

The expression
\[ -Y(\gamma_0(t_{00}) - t) f_{01} \dot{\gamma}_0(t_{00}) \delta t_0 \]

is the effect of the discontinuous initial condition (2.2) and perturbation of the initial moment \( t_{00} \).

The expression
\[ Y(\gamma_0(t_{00}) - t) f_{01} \dot{\gamma}_0(t_{00}) \delta \tau(\gamma_0(t_{00})) + \int_{t_{00}}^{t} Y(s; t) f_{0y}[s] \dot{x}_0(\tau_0(s)) \delta \tau(s) ds \]

is the effect of perturbation of the delay function \( \tau_0(t) \) (see (2.4) and (2.5)).

The addend
\[ Y(t_{00} - t) \[ \dot{\varphi}_0(t_{00}) - A(t_{00}) \dot{\varphi}_0(\sigma(t_{00})) - f_{0}^{-} \] \delta t_0 + \Psi(t_{00}; t) \[ \delta x_0 - \varphi_0(t_{00}) \delta t_0 \] \]

is the effect of perturbations of the initial moment \( t_{00} \) and the initial vector \( x_{00} \).

The expression
\[ \int_{t_{00}}^{t_{00}} Y(\gamma_0(s); t) f_{0y}[\gamma_0(s)] \dot{\gamma}_0(s) \delta \varphi(s) ds + \int_{\sigma(t_{00})}^{t_{00}} Y(\nu(s); t) A(\nu(s)) \dot{\nu}(s) \delta \varphi(s) ds \]
\[ + \int_{t_{00}}^{t} Y(s; t) \delta f[s] ds \]

is the effect of perturbations of the initial function \( \varphi_0(t) \) and the function \( f_0(t, x, y) \).

It is clear that, if \( \varphi_0(t_{00}) = x_{00} \) then \( f_{01}^{-} = 0 \).
If \( \gamma_0(t_{00}) = t_{10} \) then Theorem 2.1 is valid on the interval \([t_{10}, t_{10} + \delta_2]\). If \( \gamma_0(t_{00}) > t_{10} \) then Theorem 2.1 is valid and \( \delta_2 \in (0, \delta_1) \) is such that \( t_{10} + \delta_2 < \gamma_0(t_{00}) \), in this case \( Y(\gamma_0(t_{00})-; t) = \Theta \).

Finally we note that the variation formula allows to obtain an approximate solution of the perturbed equation

\[
\dot{x}(t) = A(t)x(t) + f_0(t, x(t), x(\tau_0(t) + \varepsilon \delta \tau(t))) + \varepsilon \delta f(t, x(t), x(\tau_0(t) + \varepsilon \delta \tau(t)))
\]

with the perturbed initial condition

\[
x(t) = \varphi_0(t) + \varepsilon \delta \varphi(t), t \in [\hat{\tau}, t_{00} + \varepsilon \delta t_0], x(t_{00} + \varepsilon \delta t_0) = x_{00} + \varepsilon \delta x_0.
\]

In fact, for a sufficiently small \( \varepsilon \in (0, \varepsilon_2] \) it follows from (2.3) that

\[
x(t; \mu_0 + \varepsilon \delta \mu) \approx x_0(t) + \varepsilon \delta x(t; \delta \mu).
\]

**THEOREM 2.2.** Let the conditions 2.1 and 2.2 of Theorem 2.2 hold. Moreover, there exist the limits

\[
\lim_{z \to z_0} f_0(z) = f_0^+, z \in [t_{00}, \gamma_0(t_{00})) \times O^2,
\]

\[
\lim_{(z_1, z_2) \to (z_{01}, z_{02})} [f_0(z_1) - f_0(z_2)] = f_{01}^+, z_i \in [\gamma_0(t_{00}), b) \times O^2, i = 1, 2.
\]

Then there exist numbers \( \varepsilon_2 \in (0, \varepsilon_1) \) and \( \delta_2 \in (0, \delta_1) \) such that for arbitrary

\[
(t, \varepsilon, \delta \mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2) \times V^+,
\]

where \( V^+ = \{ \delta \mu \in V : \delta t_0 \geq 0, \delta \tau(t_{00}) < 0 \} \), formula (2.3) is valid, where

\[
\delta x(t; \delta \mu) = \left\{ Y(t_{00}+; t) [\dot{\varphi}_0(t_{00}) - A(t_{00})\dot{\varphi}_0(\sigma(t_{00})) - f_0^+] - \right
\]

\[
Y(\gamma_0(t_{00})+; t)f_{01}^+ \gamma_0(t_{00}) \delta t_0 + Y(\gamma_0(t_{00})+; t)f_{01}^+ \gamma_0(t_{00}) \delta \tau(\gamma_0(t_{00})) + \beta(t; \delta \mu).
\]

Theorem 2.2 corresponds to the case when the variation at the point \( t_{00} \) is performed on the right.

**THEOREM 2.3.** Let the assumptions of Theorems 2.1 and 2.2 be fulfilled. Moreover,

\[
f_0^- = f_0^+ := \hat{f}_0, f_{01}^- = f_{01}^+ := \hat{f}_{01}
\]

and

\[
t_{00}, \gamma_0(t_{00}) \notin \{ \sigma(t_{10}), \sigma^2(t_{10}), \ldots \}.
\]
Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ such that for arbitrary

$$(t, \varepsilon, \delta \mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2] \times V$$

formula (2.3) holds, where

$$\delta x(t; \delta \mu) = \left\{ Y(t_{00}; t) \left[ \varphi_0(t_{00}) - A(t_{00}) \hat{\varphi}_0(\sigma(t_{00})) - \dot{f}_0 \right] - 
Y(\gamma_0(t_{00}); t) \hat{f}_0 \gamma_0(t_{00}) \right\} \delta t_0 + 
Y(\gamma_0(t_{00}); t) \hat{f}_0 \gamma_0(t_{00}) \delta \tau(\gamma_0(t_{00})) + \beta(t; \delta \mu).$$

Theorem 2.3 corresponds to the case when the variation at the point $t_{00}$ two-sided is performed. If the function $f_0(t, x, y)$ is continuous then

$$\hat{f}_0 = f_0(t_{00}, \varphi_0(t_{00}), \varphi_0(\tau_0(t_{00})))$$

and

$$\hat{f}_{01} = f_0(\gamma_0(t_{00}), x_0(\gamma_0(t_{00})), \nu(t_{00})), x_0(\gamma_0(t_{00})), \varphi_0(t_{00})).$$

**THEOREM 2.4.** Let $\gamma_0(t_{00}) < t_{10}$ and

$$f_0(t, x, y) = B(t)x + C(t)y,$$

where $B(t), C(t)$ are continuous matrix functions. Moreover,

$$t_{00} \notin \{ \sigma(t_{10}), \sigma^2(t_{10}), \ldots \}.$$

Then there exist numbers $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta_2 \in (0, \delta_1)$ such that for arbitrary

$$(t, \varepsilon, \delta \mu) \in [t_{10} - \delta_2, t_{10} + \delta_2] \times (0, \varepsilon_2] \times V$$

formula (2.3) holds, where

$$\delta x(t; \delta \mu) = \left\{ Y(t_{00}; t) \left[ \varphi_0(t_{00}) - A(t_{00}) \hat{\varphi}_0(\sigma(t_{00})) - B(t_{00}) x_0(t_{00}) - 
C(t_{00}) x_0(\tau_0(t_{00})) \right] - Y(\gamma_0(t_{00}); t) C(t_{00})(x_0(\varphi_0(t_{00})) - \varphi_0(t_{00})) \right\} \delta t_0 + 
Y(\gamma_0(t_{00}); t) C(t_{00})(x_0(\varphi_0(t_{00})) - \varphi_0(t_{00})) \delta \tau(\gamma_0(t_{00})) + \Psi(t_{00}; t) \left[ \delta x_0(t_{00}) \hat{\varphi}_0(t_{00}) \right] \right\} \delta t_0 + 
\int_{t_{00}}^{t} Y(s; t) C(s) \dot{x}_0(\tau_0(s)) \delta \tau(s) ds + \int_{\tau_0(t_{00})}^{t} Y(\gamma_0(s); t) C(\gamma_0(s)) \dot{\gamma}_0(s) \delta \varphi(s) ds + 
\int_{\sigma(t_{00})}^{t} Y(\nu(s); t) A(\nu(s)) \dot{\nu}(s) \delta \varphi(s) ds + \int_{t_{00}}^{t} Y(s; t) \delta f(s) ds;$$

$\Psi(s; t)$ and $Y(s; t)$ are matrix functions satisfying the system

$$\left\{ \begin{array}{l}
\Psi_s(s; t) = -Y(s; t) B(t) - Y(\gamma_0(s); t) C(\gamma_0(s)) \dot{\gamma}_0(s), \\
Y(s; t) = \Psi(s; t) + Y(\nu(s); t) A(\nu(s)) \dot{\nu}(s), s \in [t_{00} - \delta_2, t_{10}].
\end{array} \right.$$
REFERENCES


