PERIODIC SOLUTIONS OF NICHOLSON-TYPE DELAY DIFFERENTIAL SYSTEMS

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Abstract. In this paper a class of nonlinear nonautonomous Nicholson-type system is considered. By using the continuation theorem of coincidence degree theory, we derive a set of easily verifiable sufficient conditions that guarantee the existence of at least one positive periodic solution. By constructing Lyapunov functional, the uniqueness and global attractivity of the positive periodic solutions are obtained.

Key Words. Nicholson-type delay systems, positive periodic solutions, existence and uniqueness coincidence degree, Lyapunov functional, Barbalat’s Lemma.

AMS(MOS) subject classification. 34K20, 34K13

1. Introduction. In this paper we investigate a Nicholson-type delay differential system which is used to describe models of Marine Protected Areas and B-cell Chronic Lymphocytic Leukemia dynamics:

\[
\begin{align*}
\dot{x}_1(t) &= -a_1(t)x_1(t) + b_1(t)x_2(t) + c_1(t)x_1(t-\tau)e^{-x_1(t-\tau)}, \\
\dot{x}_2(t) &= -a_2(t)x_2(t) + b_2(t)x_1(t) + c_2(t)x_2(t-\tau)e^{-x_2(t-\tau)}.
\end{align*}
\]

Berezansky et al. [5]-[6] have studied thoroughly the dynamics of the above system with constant and time-varying parameters. It is well-known that biological or environmental parameters are subject to fluctuation in time. So the variation of the environment plays an important role in many dynamic problems of the biological and ecological models described by non-autonomous systems [9],[13]. In particular, the effects of a periodically varying environment are important

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for evolutionary theory, since the selective forces on systems in a fluctuating environment differ from those in a stable one. Thus, the assumption of periodicity of the parameters in the system incorporates the periodicity of the environment (e.g., seasonal effects of weather, food supplies, mating habits, etc.). In view of this it is realistic to assume that the parameters are periodic functions of period $\omega$.

Every basic and important ecological issue associated with the study of multispecies population interaction in periodic environments is the existence of positive periodic solutions with strictly positive components which plays the same role as the equilibrium of autonomous systems. Therefore it is natural to find out for which conditions the resulting periodic non-autonomous system would have a periodic solution.

Very interesting results on existence, positivity and stability of periodic solutions for general delay differential equations for functional differential equations and systems of these equations were studied in the following works [1], [2], [3], [7], [10], [11].

In this paper we will investigate the existence, uniqueness and stability of positive periodic solutions of the Nicholson-type system.

The paper is organized as follows: In Section 2, we introduce some notations and definitions, and state some preliminary results needed in later sections. Then we establish, in Section 3 some simple criteria for the existence of positive periodic solutions of the system by using the continuation theorem and coincidence degree theory [8]. The uniqueness and global attractivity of the positive periodic solution are presented in Section 4.

2. Preliminaries. Let us consider the following Nicholson-type system

$$\begin{align*}
\dot{x}_1(t) &= -a_1(t)x_1(t) + b_1(t)x_2(t) + c_1(t)x_1(t-\tau)e^{-x_1(t-\tau)} \\
\dot{x}_2(t) &= -a_2(t)x_2(t) + b_2(t)x_1(t) + c_2(t)x_2(t-\tau)e^{-x_2(t-\tau)},
\end{align*}$$

where

$(a_1) \ a_i, b_i, c_i, i = 1, 2, \tau$ are positive, continuous and $\omega$-periodic functions, $\omega > 0$;

$(a_2) \ x_i(0) > 0$, for $i = 1, 2$.

Particularly if $f$ is periodic of period $\omega$, we shall denote the average of $f$ on the interval $[0, \omega]$ by

$$\bar{f} = \frac{1}{\omega} \int_{0}^{\omega} f(t)dt.$$ 

In addition let us define:

$$f^- = \min_{t \in [0, \omega]} f(t), \ f^+ = \max_{t \in [0, \omega]} f(t).$$
In Section 3 we will use the continuation theorem of coincidence degree theory to establish criteria for the existence of at least one positive $\omega$-periodic solution of system (1). To this purpose we will summarize a few concepts and results from [8] that will be basic for Section 3.

**Definition 1.** [18, 19]

Let $X$ and $Y$ be two Banach spaces, $L : \text{Dom} L \subset X \to Y$ be a linear mapping. The mapping $L$ will be called a Fredholm mapping of index zero if the following three conditions hold:

(i) $\text{Ker} L$ has a finite dimension;
(ii) $\text{Im} L$ is closed and has a finite codimension;
(iii) $\text{Dim Ker} L = \text{codim Im} L < \infty$.

If $L$ is a Fredholm mapping of index zero and there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$, such that $\text{Im} P = \text{Ker} L, \text{Im} g L = \text{Ker} Q = \text{Im} (I − Q)$, it follows that

$$L|\text{Dom} L \cap \text{Ker} P : (I − P)X \to \text{Im} L$$

is invertible. We denote the inverse of that map by $K_p$.

**Definition 2.** [8] If $\Omega$ is an open bounded subset of $X$, the mapping $N$ will be called $L$-compact on $\Omega$ if the mapping $QN : \Omega \to Y$ is continuous, $QN(\Omega)$ is bounded, and $K_p(I − Q)N : \Omega \to X$ is compact, i.e., it is continuous and $K_p(I − Q)N(\Omega)$ is relatively compact, where

$$K_p : \text{Im} L \to \text{Dom} L \cap \text{Ker} P$$

is the inverse of the restriction $L_p$ of $L$ to $\text{Dom} L \cap \text{Ker} P$, so that $LK_p = I$ and $K_p L = I − P$. Since $Q$ is isomorphic to $\text{Ker} L$, there exists an isomorphic $J : \text{Im} Q \to \text{Ker} L$.

**Lemma 1.** (Continuation Theorem)[8] Let $X$ and $Y$ be two Banach spaces and $L$ a Fredholm mapping of index zero. Assume that $N : \Omega \to Y$ is $L$-compact on $\Omega$ with $\Omega$ is open bounded subset of $X$. Assume there exists continuous projector $Q : Y \to Y$. Furthermore assume:

(a) for each $\lambda \in (0, 1)$, every solution of $Lx = \lambda Nx$ is such that $x \notin \partial \Omega$;
(b) $QN x \neq 0$ for each $x \in \partial \Omega \cap \text{Ker} L$ and

$$\text{deg} \{QN x, \Omega \cap \text{Ker} L, 0\} \neq 0.$$  

Then the operator equation $Lx = Nx$ has at least one solution in $\text{Dom} (L \cap \Omega)$. 
3. Existence of periodic solutions. In this section we establish sufficient conditions for the existence of \( \omega \)-positive periodic solutions of (1).

We denote

\[
A_1 = 2 \int_0^\omega a_1(t) dt, \quad A_2 = 2 \int_0^\omega a_2(t) dt
\]

\[
B_1 = \int_0^\omega b_1(t) dt, \quad B_2 = \int_0^\omega b_2(t) dt
\]

\[
C_1 = \int_0^\omega c_1(t) dt, \quad C_2 = \int_0^\omega c_2(t) dt
\]

\[
D_i = \max \{B_i, C_i\}, i = 1, 2.
\]

**Theorem 1.** Assume that \((a_1) - (a_2)\) hold and

\[
2D_{1\text{min}} \{e^{A_1}, e^{A_2}\} < A_1 \leq 4D_{1\text{max}} \{e^{A_1}, e^{A_2}\}
\]

\[
2D_{2\text{min}} \{e^{A_1}, e^{A_2}\} < A_2 \leq 4D_{2\text{max}} \{e^{A_1}, e^{A_2}\}
\]

\[
2C_i > e^{A_i} A_i, i = 1, 2.
\]

Then system (1) has at least one positive \( \omega \)-periodic solution \( x^*(t) = \{x_1^*(t), x_2^*(t)\}^T \) with strictly positive components, and there exist positive constants \( \alpha_i, \beta_i \) such that \( \alpha_i \leq x_i^* \leq \beta_i \).

**Proof.** Let \( x_i(t) = e^{u_i(t)}, i = 1, 2. \) on substituting into (1), we obtain:

\[
\begin{align*}
\dot{u}_1(t) &= -a_1(t) + b_1(t)e^{u_1(t)-u_2(t)} + c_1(t)e^{u_1(t)-u_1(t)-e^{u_1(t-\tau)}} \\
\dot{u}_2(t) &= -a_2(t) + b_2(t)e^{u_2(t)-u_1(t)} + c_2(t)e^{u_2(t-\tau)-u_2(t)-e^{u_2(t-\tau)}}.
\end{align*}
\]

In order to apply Lemma 2.1 we first define:

\[U = Z = \{u(t) = (u_1(t), u_2(t))^T \in C(R, R^2) : u(t + \omega) = u(t)\}\]

and

\[\|U\| = \|(u_1(t), u_2(t))^T\| = \max_{t \in [0, \omega]} |u_1(t)| + \max_{t \in [0, \omega]} |u_2(t)|, u_i \in U \text{ (or } Z).\]

Then \( U \) and \( Z \) are Banach spaces with the norm they are endowed \( \|.\|. \). Let

\[
Nu = \begin{pmatrix}
-a_1(t) + b_1(t)e^{u_1(t)-u_2(t)} + c_1(t)e^{u_1(t-\tau)-u_1(t)-e^{u_1(t-\tau)}} \\
-a_2(t) + b_2(t)e^{u_2(t)-u_1(t)} + c_2(t)e^{u_2(t-\tau)-u_2(t)-e^{u_2(t-\tau)}}
\end{pmatrix}
\]
Then and
\[ Lu = u' = \left[ \frac{du_1}{dt} \frac{du_2}{dt} \right] , \]
\[ Pu = \left[ \frac{1}{\omega} \int_0^\omega u_1(t)dt \int_0^\omega u_2(t)dt \right] , u \in U, \]
\[ Qz = \left[ \frac{1}{\omega} \int_0^\omega z_1(t)dt \int_0^\omega z_2(t)dt \right] , z \in Z. \]

Then,
\[ KerL = \{(u_1(t), u_2(t))^T \in U : (u_1(t), u_2(t))^T = (h_1, h_2) \in \mathbb{R}^2\}, \]
\[ ImL = \left\{ z = (u_1(t), u_2(t))^T \in Z : \int_0^w z(t)dt = 0 \right\} \]
is closed in \( Z \), and \( \dim KerL = 2 = \text{codim} ImL \). Then \( L \) is a Fredholm mapping of index zero. Since \( ImL \) is closed in \( Z \), and \( L \) is a Fredholm mapping of index zero, it is easy to show that \( P \) and \( Q \) are continuous projectors such that \( ImP = KerL, KerQ = ImL = Im(I - Q). \)

Furthermore, the generalized inverse (of \( L \))
\[ KP : ImL \to KerP \cap DomL \]
exists, which is given by
\[ KP(z) = \int_0^t z(s)ds - \frac{1}{\omega} \int_0^\omega \int_0^t z(s)dsdt. \]

Then
\[ QN : U \to Z, KP(I - Q) : U \to U \]
\[ QN \left[ \begin{array}{c} u_1 \\ u_2 \end{array} \right] = \left( \begin{array}{c} \frac{1}{\omega} \int_0^\omega -a_1(t) + b_1(t)e^{u_1(t)-u_2(t)} + c_1(t)e^{u_1(t-t)} - u_1(t) - e^{u_1(t-t)} \\ \frac{1}{\omega} \int_0^\omega -a_2(t) + b_2(t)e^{u_2(t)-u_1(t)} + c_2(t)e^{u_2(t-t)} - u_2(t) - e^{u_2(t-t)} \end{array} \right), \]
\[ KP(I - Q)N \left[ \begin{array}{c} u_1 \\ u_2 \end{array} \right] = \left( \begin{array}{c} \int_0^t -a_1(s) + b_1(s)e^{u_1(s)-u_2(s)} + c_1(s)e^{u_1(s-s)} - u_1(s) - e^{u_1(s-s)} ds \\ \int_0^t -a_2(s) + b_2(s)e^{u_2(s)-u_1(s)} + c_2(s)e^{u_2(s-s)} - u_2(s) - e^{u_2(s-s)} ds \end{array} \right) - \left( \begin{array}{c} \frac{1}{\omega} \int_0^\omega \int_0^t -a_1(s) + b_1(s)e^{u_1(s)-u_2(s)} + c_1(s)e^{u_1(s-s)} - u_1(s) - e^{u_1(s-s)} ds \\ \frac{1}{\omega} \int_0^\omega \int_0^t -a_2(s) + b_2(s)e^{u_2(s)-u_1(s)} + c_2(s)e^{u_2(s-s)} - u_2(s) - e^{u_2(s-s)} ds \end{array} \right). \]
\(- \left( \frac{r}{\omega} - \frac{1}{2} \right) \int_0^\omega a_1(s) + b_1(s)e^{u_1(s)-u_2(s)} + c_1(s)e^{u_1(s-\tau)-u_1(s)-e^{u_1(s-\tau)}} ds \right) \cdot \left( \frac{r}{\omega} - \frac{1}{2} \right) \int_0^\omega a_2(s) + b_2(s)e^{u_2(s)-u_1(s)} + c_2(s)e^{u_2(s-\tau)-u_2(s)-e^{u_2(s-\tau)}} ds \right). \]

The integration form of the terms of both \(QN\) and \(K_P(I - Q)N\) implies that they map continuously differentiable with respect to \(t\) and they are map bounded continuous functions to the bounded continuous functions. By the Ascoli-Arzela Theorem [12], we see that \(QN(\Omega)\) and \(K_P(I - Q)N(\Omega)\) are relatively compact for any bounded set \(\Omega \subset X\). Thus, \(N\) is \(L\)-compact on \(\Omega\) for any open bounded set \(\Omega \subset X\). Now we are ready to the proof of Lemma 2.1. Corresponding to the operator equation \(L_x = \lambda N_x, \lambda \in (0, 1)\), we have

\[
\begin{align*}
\dot{u}_1(t) &= \lambda \left\{ -a_1(t) + b_1(t)e^{u_1(t)-u_2(t)} + c_1(t)e^{u_1(t-\tau)-u_1(t)-e^{u_1(t-\tau)}} \right\} \\
\dot{u}_2(t) &= \lambda \left\{ -a_2(t)b_2(t)e^{u_2(t)-u_1(t)} + c_2(t)e^{u_2(t-\tau)-u_2(t)-e^{u_2(t-\tau)}} \right\}.
\end{align*}
\]

Assume that \(u(t) = (u_1(t), u_2(t))^T \in X\) is a solution of (3) for some \(\lambda \in (0, 1)\). Then there exist \(\zeta_i, \eta_i \in [0, \omega]\) such that

\[
\begin{align*}
u_i(\zeta_i) &= \min_{t \in [0, \omega]} u_i(t), \\u_i(\eta_i) &= \max_{t \in [0, \omega]} u_i(t), \\i = 1, 2, \dot{u}_i(\zeta_i) = \dot{u}_i(\eta_i) = 0.
\end{align*}
\]

Integrating (3) over the interval \([0, \omega]\) leads to

\[
\int_0^\omega a_1(t)dt = \int_0^\omega \left\{ b_1(t)e^{u_1(t)-u_2(t)} + c_1(t)e^{u_1(t-\tau)-u_1(t)-e^{u_1(t-\tau)}} \right\} dt.
\]

Then we get

\[
\begin{align*}
\int_0^\omega |\dot{u}_1(t)| dt &\leq \lambda \int_0^\omega a_1(t)dt \\
&+ \lambda \int_0^\omega \left\{ b_1(t)e^{u_1(t)-u_2(t)} + c_1(t)e^{u_1(t-\tau)-u_1(t)-e^{u_1(t-\tau)}} \right\} dt \\
&< 2 \int_0^\omega a_1(t)dt = A_1.
\end{align*}
\]

In the same way, we obtain

\[
\int_0^\omega |\dot{u}_2(t)| dt < 2 \int_0^\omega a_2(t)dt = A_2.
\]

Consider now 4 cases

a) If \(u_2(\xi_2) > u_1(\xi_1)\) and \(u_1(\eta_1) > u_2(\eta_2)\) then

\[
\frac{A_1}{2} = \int_0^\omega a_1(t)dt \leq e^{u_1(\eta_1)-u_2(\xi_2)} \int_0^\omega b_1(t)dt + e^{u_1(\eta_1)-u_1(\xi_1)-e^{u_1(\xi_1)}} \int_0^\omega c_1(t)dt
\]
\[ \leq D_1 e^{u_1(\eta_1) - u_1(\xi_1)} (1 + e^{-e^{u_1(\xi_1)}}). \]

Now
\[ \frac{(A_1/2D_1)e^{-u_1(\eta_1)}}{1 + e^{-e^{u_1(\xi_1)}}} \leq e^{-u_1(\xi_1)}, \]
then we obtain
\[ u_1(\xi_1) \leq \ln \left( \frac{1 + e^{-e^{u_1(\xi_1)}}}{(A_1/2D_1)e^{-u_1(\eta_1)}} \right). \]
Hence
\[ u_1(\xi_1) \leq \ln(1 + e^{-e^{u_1(\xi_1)}}) - \ln(A_1/2D_1) + u_1(\eta_1). \]

Now
\[ u_1(t) \leq u_1(\xi_1) + \int_0^\omega |\dot{u}_1(t)|\,dt \leq \ln(1 + e^{-e^{u_1(\xi_1)}}) - \ln(A_1/2D_1) + u_1(\eta_1) + A_1. \]
In particular
\[ u_1(\eta_1) \leq \ln(1 + e^{-e^{u_1(\xi_1)}}) - \ln(A_1/2D_1) + u_1(\eta_1) + A_1. \]
Hence
\[ \ln(1 + e^{-e^{u_1(\xi_1)}}) \geq \ln(A_1/2D_1) - A_1. \]
Then
\[ e^{-e^{u_1(\xi_1)}} \geq \frac{A_1}{2D_1e^{A_1}} - 1, \]
therefore we get
\[ -e^{u_1(\xi_1)} \geq \ln\left( \frac{A_1}{2D_1e^{A_1}} - 1 \right). \]
Which leads to
\[ u_1(\xi_1) \leq \ln\left( \ln\left( \frac{2D_1e^{A_1}}{A_1 - 2D_1e^{A_1}} \right) \right), \]
hence
\[ (8) \quad u_1(t) \leq \ln\left( \ln\left( \frac{2D_1e^{A_1}}{A_1 - 2D_1e^{A_1}} \right) \right) + A_1 =: H_1. \]
Now if we evaluate by (4)-(5)
\[ \frac{A_2}{2} = \int_0^\omega a_2(t)\,dt \leq e^{u_1(\eta_1) - u_1(\xi_1)} \int_0^\omega b_2(t)\,dt + e^{u_1(\eta_1) - u_1(\xi_1) - e^{u_1(\xi_1)}} \int_0^\omega c_2(t)\,dt \leq D_2 e^{u_1(\eta_1) - u_1(\xi_1)} (1 + e^{-e^{u_1(\xi_1)}}). \]
Hence we obtain similarly

\[ u_1(t) \leq \ln \left( \ln \left( \frac{2D_2 e^{A_1}}{A_2 - 2D_2 e^{A_1}} \right) \right) + A_1 =: H_2. \]

b) If \( u_2(\xi_2) \leq u_1(\xi_1) \) and \( u_1(\eta_1) \leq u_2(\eta_2) \) then

\[ \frac{A_2}{2} = \int_0^\omega a_2(t)dt \leq \int_0^\omega b_2(t)dt + \int_0^\omega c_2(t)dt \]

\[ \leq D_2 e^{u_2(\eta_2) - u_2(\xi_2)} (1 + e^{-u_2(\xi_2)}), \]

Hence we get similarly

\[ u_2(t) \leq \ln \left( \ln \left( \frac{2D_2 e^{A_2}}{A_2 - 2D_2 e^{A_2}} \right) \right) + A_2 =: H_3, \]

and if we evaluate

\[ \frac{A_1}{2} = \int_0^\omega a_1(t)dt \leq \int_0^\omega b_1(t)dt + \int_0^\omega c_1(t)dt \]

\[ \leq 2D_1 e^{u_2(\eta_2) - u_2(\xi_2)} (1 + e^{-u_2(\xi_2)}), \]

then

\[ u_2(t) \leq \ln \left( \ln \left( \frac{2D_1 e^{A_2}}{A_1 - 2D_1 e^{A_2}} \right) \right) + A_2 =: H_4. \]

c) If \( u_2(\xi_2) > u_1(\xi_1) \) and \( u_1(\eta_1) < u_2(\eta_2) \) then by using the fact that \( \max_{u \geq 0} u e^{-u} = 1/e \),

\[ \frac{A_1}{2} = \int_0^\omega a_1(t)dt \leq e^{u_1(\eta_1) - u_1(\xi_1)} \int_0^\omega b_1(t)dt + e^{-u_1(\xi_1)} \int_0^\omega c_1(t)dt \]

\[ \leq D_1 e^{-u_1(\xi_1)} (e^{u_1(\eta_1)} + 1). \]

Hence

\[ e^{u_1(\xi_1)} \leq \frac{2D_1 (e^{u_1(\eta_1)} + 1)}{A_1}. \]
By taking logarithm from both sides we obtain
\[ u_1(\xi_1) \leq \ln \left( \frac{2D_1(e^{u_1(\eta_1)} + 1)}{A_1} \right). \]

By using the fact that \( \ln(x + h) < \ln(x) + \frac{h}{x}, x > 0 \) we obtain
\[ u_1(\xi_1) \leq \ln \left( \frac{2D_1}{A_1} \right) + u_1(\eta_1) + e^{-u_1(\eta_1)}. \]

Now
\[ u_1(t) \leq u_1(\xi_1) + \int_0^\omega |\dot{u}_1(t)|dt \leq \ln \left( \frac{2D_1}{A_1} \right) + u_1(\eta_1) + e^{-u_1(\eta_1)} + A_1. \]

In particular
\[ u_1(\eta_1) \leq u_1(\xi_1) + \int_0^\omega |\dot{u}_1(t)|dt \leq \ln \left( \frac{2D_1}{A_1} \right) + u_1(\eta_1) + e^{-u_1(\eta_1)} + A_1. \]

Hence
\[ e^{-u_1(\eta_1)} \geq \ln \left( \frac{A_1}{2D_1} \right) - A_1. \]

Taking logarithm from both sides, we obtain
\[ u_1(\eta_1) \leq \ln \left( \frac{1}{\ln \left( \frac{A_1}{2D_1} \right) - A_1} \right). \]

Hence
\[ u_1(t) \leq \ln \left( \frac{1}{\ln \left( \frac{A_1}{2D_1} \right) - A_1} \right) := H_5. \]

d) If \( u_2(\xi_2) < u_1(\xi_1) \) and \( u_1(\eta_1) > u_2(\eta_2) \) then
\[ \frac{A_2}{2} = \int_0^\omega a_2(t)dt \leq e^{u_2(\eta_2) - u_2(\xi_2)} \int_0^\omega b_2(t)dt + e^{-u_2(\xi_2)} \int_0^\omega c_2(t)dt \]
\[ \leq D_2 e^{-u_2(\xi_2)} (e^{u_2(\eta_2)} + 1). \]
Hence the upper bound is obtained similarly for \( u_2(t) \)

\[
u_2(t) \leq \ln \left( \frac{1}{\ln \left( \frac{A_2}{2D_2} \right) - A_2} \right) := H_6.
\]

Then finally we obtain

\[
u_1(t) \leq \max \{H_1, H_2, H_5\} = \bar{H}_1,
\]

\[
u_2(t) \leq \max \{H_3, H_4, H_6\} = \bar{H}_2.
\]

Now let us evaluate \( u_i(t), i = 1, 2 \) from below.

\[
\frac{A_1}{2} = \int_0^\omega a_1(t)dt = e^{u_1(t)-u_2(t)} \int_0^\omega b_1(t)dt + e^{u_1(t_\tau)-u_1(t)-e^{u_1(t_\tau-t)}} \int_0^\omega c_1(t)dt
\]

\[
\geq e^{u_1(\xi_1)-u_2(\eta_2)} \int_0^\omega b_1(t)dt + e^{u_1(\xi_1)-u_1(\eta_1)-e^{u_1(\eta_1)}} \int_0^\omega c_1(t)dt \geq C_1 e^{u_1(\xi_1)-u_1(\eta_1)-e^{u_1(\eta_1)}}
\]

then

\[
\frac{A_1}{2C_1} e^{-u_1(\xi_1)} \geq e^{-u_1(\eta_1)-e^{u_1(\eta_1)}}.
\]

So we obtain

\[
\ln \left( \frac{A_1}{2C_1} \right) - u_1(\xi_1) \geq -u_1(\eta_1) - e^{u_1(\eta_1)}.
\]

Hence

\[
u_1(\eta_1) \geq u_1(\xi_1) + \ln \left( \frac{2C_1}{A_1} \right) - e^{u_1(\eta_1)}.
\]

Since

\[
u_1(t) \geq u_1(\eta_1) - A_1 \geq u_1(\xi_1) + \ln \left( \frac{2C_1}{A_1} \right) - e^{u_1(\eta_1)} - A_1,
\]

in particular

\[
u_1(\eta_1) \geq u_1(\eta_1) - A_1 \geq u_1(\xi_1) + \ln \left( \frac{2C_1}{A_1} \right) - e^{u_1(\eta_1)} - A_1,
\]

hence

\[
e^{u_1(\eta_1)} \geq \ln \left( \frac{2C_1}{A_1} \right) - A_1.
\]

Then

\[
u_1(\eta_1) \geq \ln \left( \ln \left( \frac{2C_1}{A_1} \right) - A_1 \right).
\]
Finally we obtain

\begin{equation}
 u_1(t) \geq \ln \left( \ln \left( \frac{2C_1}{A_1} \right) - A_1 \right) - A_1 = \bar{H}_3.
\end{equation}

Similarly

\begin{equation}
 u_2(t) \geq \ln \left( \ln \left( \frac{2C_2}{A_2} \right) - A_2 \right) - A_2 = \bar{H}_4.
\end{equation}

Let $H > \max \{ |\bar{H}_1|, |\bar{H}_2|, |\bar{H}_3|, |\bar{H}_4| \}$ be a fix constant and define

\[ \Omega := \{ u(t) = (u_1(t), u_2(t))^T \in U : \| u \| < H \} \]

Then (8), (9), (10), (11), (12), (13) imply that there is no $\lambda \in (0, 1)$ and $u \in \partial \Omega$ such that $Lu = Nu$.

If $u(t) = (u_1(t), u_2(t))^T \in \partial \Omega \cap \text{Ker} L$ then $u(t)$ is a constant vector in $R^2$, and there exists some $i \in 1, 2$, such that $|u_i| = H$. Assume $|u_1| = H$, so that $u_1 = H$. Then, we claim

\begin{equation}
 (QN(u))_1 > 0 \text{ for } u_1 = -H, \text{ and } (QN(u))_1 < 0 \text{ for } u_1 = H.
\end{equation}

If $(QN(u))_1 \leq 0$ for $u_1 = -H$ it follows from that

\[ \frac{1}{\omega} \int_0^\omega -a_1(t) + b_1(t)e^{u_1(t)} - u_2(t) + c_1(t)e^{u_1(t)} - u_1(t)e^{u_1(t)} \ dt \leq 0. \]

Hence

\[ \frac{A_1}{2} = \int_0^\omega a_1(t) \geq \int_0^\omega b_1(t)e^{u_1(t)} - u_2(t) + c_1(t)e^{u_1(t)} - u_1(t)e^{u_1(t)} \ dt \]

\[ \geq e^{-e^{-H}} \int_0^\omega c_1(t) \ dt = C_1 e^{-e^{-H}}, \]

which yields

\[ e^{-e^{-H}} \leq \frac{A_1}{2C_1}. \]

Then

\[ e^{-H} \geq \ln \left( \frac{2C_1}{A_1} \right) \]

or

\[ -H \geq \ln \left( \ln \left( \frac{2C_1}{A_1} \right) \right) \geq \ln \left( \ln \left( \frac{2C_1}{A_1} \right) - A_1 \right) \geq \ln \left( \ln \left( \frac{2C_1}{A_1} \right) - A_1 \right) - A_1 = \bar{H}_3. \]
This is a contradiction which implies \((QN(u))_1 > 0\) for \(u_1 = -H\).

Similarly, if \((QN(u))_1 \geq 0\), for \(u_1 = H\), it follows that
\[
\frac{A_1}{2} = \int_0^\infty a_1(t) \leq e^{H-U_2(\xi_2)} \int_0^\infty b_1(t) + e^{-e^H} \int_0^\infty c_1(t) dt.
\]
Since \(H = u_1(\xi)\) and \(U_2(\xi_2) > u_1(\xi)\) then \(U_2(\xi_2) > H\).

Hence
\[
\frac{A_1}{2} \leq B_1 + e^{-e^H} C_1 \leq D_1(1 + e^{-e^H}).
\]

Then
\[
e^{-e^H} \geq \frac{A_1}{2D_1} - 1,
\]

finally we obtain
\[
H \leq \ln \left( \ln \left( \frac{2D_1}{A_1 - 2D_1} \right) \right) \leq \ln \left( \ln \left( \frac{2D_1 e^{A_1}}{A_1 - 2D_1 e^{A_1}} \right) \right) + A_1 = H_1,
\]

which is a contradiction to the choice of \(H\), thus \((QN(u))_1 < 0\), for \(u_1 = H\).

Similarly, if \(|u_2| = H\), we obtain \((QN(u))_2 > 0\) for \(u_2 = -H\), and \((QN(u))_2 < 0\) for \(u_2 = H\). Furthermore, let \(0 < \mu < 1\) and define a continuous function \(H(u, \mu)\) by setting
\[
H(u, \mu) = -(1 - \mu)u + \mu QNu.
\]

It follows from (14) that \(H(u, \mu) \neq (0, 0)^T\) for all \(u \in \partial \Omega \cap KerL\). Hence, using the homotopy invariance theorem, we obtain
\[
\text{deg} \left\{ QN, \Omega \cap kerL, (0, 0)^T \right\} = \text{deg} \left\{ -u, \partial \Omega \cap kerL, (0, 0)^T \right\} \neq 0.
\]

It then follows from the continuation theorem that \(Lx = Nx\) has a solution
\[
u^*(t) = (u_1^*, u_2^*)^T \in \text{Dom}(L \cap \overline{\Omega}),
\]
which is a \(T\)-periodic solution of system (2). Therefore
\[
x^*(t) = (x_1^*, x_2^*)^T = (e^{u_1^*}, e^{u_2^*})^T
\]
is a positive \(T\)-periodic solution of (1) and the proof is complete. 

4. Uniqueness and global attractivity of periodic solutions.

In this section we will derive the global asymptotic stability of the positive \( \omega \)-periodic solution of system (1) and deduce the uniqueness of the solution.

First we will start with a definition.

**Definition 3.** [17]

Let \((x^*_1(t), x^*_2(t))^T\) be a positive \( \omega \)-periodic solution of system (1) with positive initial data. We say that \((x^*_1(t), x^*_2(t))^T\) is globally asymptotically stable if any other positive solutions \((x_1(t), x_2(t))^T\) of (1) have the property

\[
\lim_{t \to \infty} (|x_1(t) - x^*_1(t)| + |x_2(t) - x^*_2(t)|) = 0
\]

It is clear that if \((x^*_1(t), x^*_2(t))^T\) is globally asymptotically stable, then \((x^*_1(t), x^*_2(t))^T\) is unique.

By Theorem 3.1 there exists \( T > 0 \) such that \( m_i \leq x_i \leq M_i \) for \( i = 1, 2 \) and for all \( t \geq T \).

Now, we will state our main result of the global asymptotic stability of the unique positive solution of system (1).

**Theorem 2.** Assume that \( (a_1) - (a_2) \) hold and \( c_i^- > a_i^+ e^{M_i}, c_i^+ < (a_i^- - b_i^+) e^{2}, i = 1, 2 \). Then system (1) has a unique \( \omega \)-periodic solution which is globally asymptotically stable and \( \alpha_i \leq x_i(t) \leq \beta_i \) for \( i = 1, 2 \).

**Proof.** Theorem 3.1 implies that system (1) has at least one \( \omega \)-periodic solution \((x^*_1(t), x^*_2(t))^T\), and there exist positive constants \( \alpha_i \) and \( \beta_i \) such that \( \alpha_i \leq x_i(t) \leq \beta_i \) for \( i = 1, 2 \).

To complete the proof, we only need to show that \((x^*_1(t), x^*_2(t))^T\) is globally asymptotically stable. Let \((x_1(t), x_2(t))^T\) be any other solution of (1) with initial value \((x_1(0), x_2(0))^T\).

For all \( t \geq T \) consider a Lyapunov function defined by

\[
V(t) = |\ln x_1(t) - \ln x^*_1(t)| + |\ln x_2(t) - \ln x^*_2(t)|.
\]

Calculating the upper-right derivative of \( V(t) \) along the solution of (1) leads to

\[
D^+ V(t) = \sum_{i=1}^{2} \left( \frac{x'_i(t)}{x_i(t)} - \frac{x^*_i(t)}{x^*_i(t)} \right) \text{sign}(x_i(t) - x^*_i(t)) = \text{sign}(x_1(t) - x^*_1(t))
\]

\[
\times \left( -a_1(t)x_1(t) + b_1(t)x_2(t) + c_1(t)x_1(t - \tau)e^{-x_1(t-\tau)} \right) / x_1(t)
\]
\[ D_1(t) = \frac{-a_1(t)x_1(t) + b_1(t)x_2(t) + c_1(t)x_1(t - \tau)e^{-x_1(t-\tau)}}{x_1(t)} + \frac{a_1(t)x_1^*(t) - b_1(t)x_2^*(t) - c_1(t)x_1^*(t - \tau)e^{-x_1^*(t-\tau)}}{x_1^*(t)}. \]

Otherwise if \( x_1(t) < x_1^*(t) \)

\[ D_1(t) = \frac{-a_1(t)x_1(t) + b_1(t)x_2(t) + c_1(t)x_1^*(t - \tau)e^{-x_1(t-\tau)}}{x_1(t)} + \frac{a_1(t)x_1(t) - b_1(t)x_2^*(t) - c_1(t)x_1(t - \tau)e^{-x_1(t-\tau)}}{x_1(t)}. \]

Similarly for \( D_2(t) \), if \( x_2(t) \geq x_2^*(t) \)

\[ D_2(t) = \frac{-a_2(t)x_2(t) + b_2(t)x_1(t) + c_2(t)x_2(t - \tau)e^{-x_2(t-\tau)}}{x_2(t)} + \frac{a_2(t)x_2^*(t) - b_2(t)x_1^*(t) - c_2(t)x_2^*(t - \tau)e^{-x_2^*(t-\tau)}}{x_2^*(t)}. \]

Otherwise if \( x_2(t) < x_2^*(t) \)

\[ D_2(t) = \frac{-a_2(t)x_2(t) + b_2(t)x_1(t) + c_2(t)x_2^*(t - \tau)e^{-x_2^*(t-\tau)}}{x_2^*(t)} + \frac{a_2(t)x_2(t) - b_2(t)x_1^*(t) - c_2(t)x_2(t - \tau)e^{-x_2(t-\tau)}}{x_2(t)}. \]

Now we will estimate \( D_1(t) \) under the following two cases:

(i) If \( x_1(t) \geq x_1^*(t) \), then

\[-a_1(t)x_1(t) + b_1(t)x_2(t) + c_1(t)x_1(t - \tau)e^{-x_1(t-\tau)} \geq -a_1^*m_1 + c_1^*m_1e^{-M_1} > 0.\]
Hence
\[ D_1(t) \leq -\frac{a_1^- |x_1(t) - x_1^*(t)| + b_1^+ |x_2(t) - x_2^*(t)| + \frac{c_1^+}{2} |x_1(t - \tau) - x_1^*(t - \tau)|}{\alpha_1}. \]

(ii) If \( x_1(t) \leq x_1^*(t) \), then
\[ D_1(t) \leq -\frac{a_1^- |x_1(t) - x_1^*(t)| + b_1^+ |x_2(t) - x_2^*(t)| + \frac{c_1^+}{2} |x_1(t - \tau) - x_1^*(t - \tau)|}{m_1}. \]

Similarly we obtain for \( D_2(t) \)
(i) If \( x_2(t) \geq x_2^*(t) \), then
\[-a_2(t)x_2(t) + b_1(t)x_1(t) + c_2(t)x_2(t - \tau)e^{-x_2(t-\tau)} \geq -a_2^+ m_2 + c_2^- m_2 e^{-M_2} > 0.\]
Hence
\[ D_2(t) \leq -\frac{a_2^- |x_2(t) - x_2^*(t)| + b_2^+ |x_1(t) - x_1^*(t)| + \frac{c_2^+}{2} |x_2(t - \tau) - x_2^*(t - \tau)|}{\alpha_2}. \]

(ii) If \( x_2(t) \leq x_2^*(t) \), then
\[ D_2(t) \leq -\frac{a_2^- |x_2(t) - x_2^*(t)| + b_2^+ |x_1(t) - x_1^*(t)| + \frac{c_2^+}{2} |x_2(t - \tau) - x_2^*(t - \tau)|}{m_2}. \]

Then
\[
D^+ V(t) \leq \delta (|x_1(t) - x_1^*(t)| + |x_2(t) - x_2^*(t)|
+ |x_1(t - \tau) - x_1^*(t - \tau)| + |x_2(t - \tau) - x_2^*(t - \tau)|),
\]
where
\[ \delta = \frac{1}{\max \{m_1, m_2, \alpha_1, \alpha_2\}} \min \left\{ a_1^- - b_1^+ - \frac{c_1^+}{e^2}, a_2^- - b_2^+ - \frac{c_2^+}{e^2} \right\} > 0. \]

Since the solutions of (1) are bounded then we have
\[ V(T) = |ln x_1(T) - ln x_1^*(T)| + |ln x_2(T) - ln x_2^*(T)| < \infty. \]

Integrating (14) we get, \( t \geq T \),
\[
V(t) + \delta \int_T^t (|x_1(s) - x_1^*(s)| + |x_2(s) - x_2^*(s)| + |x_1(s - \tau) - x_1^*(s - \tau)| +
\]
|x_2(s - \tau) - x_2^*(s - \tau)|ds \leq V(T) < \infty.

Then
\[
\int_T^\infty (|x_1(s) - x_1^*(s)| + |x_2(s) - x_2^*(s)| + |x_1(s - \tau) - x_1^*(s - \tau)| \\
+ |x_2(s - \tau) - x_2^*(s - \tau)|)ds \leq \frac{V(T)}{\delta} < \infty
\]

Then by Barbalat Lemma [4]
\[
limit_{t \to \infty} (|x_1(s) - x_1^*(s)| + |x_2(s) - x_2^*(s)|) = 0.
\]

The proof is complete. \qed
REFERENCES


