POPOV-TYPE STABILITY CRITERION
FOR THE FUNCTIONAL-DIFFERENTIAL EQUATIONS
DESCRIBING PULSE MODULATED CONTROL SYSTEMS

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Abstract

We study the stability of equilibrium states of nonlinear functional-differential equations describing pulse modulated control systems. The frequency-domain criteria of the Popov type are obtained.

1. Introduction

In the past three decades, a lot of effort has been devoted to the study of sample-data systems such that their mathematical description can be reduced to discrete-time models. However, numerous pulse-modulated systems of theoretical and practical interest do not admit a discrete-time reduction. The systems of this kind involve a modulated parameter (e.g. frequency, width, amplitude, phase) which depends nonlinearly on a modulated input function. So their study leads to functional-differential or functional-integral equations.

The present communication is concerned with the stability of the equilibrium states of nonlinear closed-loop pulse-modulated systems. It follows in the footsteps of papers [1]–[3]. The approach based on averaging of a modulator output function and on the Yakubovich-Kalman lemma is proposed for solving the problem. The frequency conditions obtained generalize the classical Popov stability criterion. The method proposed here resembles the equivalent areas in principle but avoids the mathematical non-rigorism of the latter. It is applicable to all kinds of modulation where an upper bound of a sampling period is known. The advantage of the present approach is that the form of the impulse may not be known exactly, so the criteria provided guarantee good robustness to variations of the modulator’s parameters.

2. Problem Formulation

The main element of a pulse-modulated system is the modulator. It is described with a nonlinear operator \( M \) which transforms a continuous output function \( \sigma(t) \) into a piecewise continuous output function \( f(t) \):

\[
    f = M\sigma
\]
The operator $M$ satisfies the causality condition: the value of $f$ at a point $t$ depends on values of $\sigma$ at preceding points $\tau$, $\tau \leq t$, only. The description of $f(t)$ involves an increasing sequence of sampling moments $t_0, t_1, t_2 \ldots (t_n \to +\infty$ as $n \to +\infty)$. When $t_n < t < t_{n+1}$ the function $f(t)$ presents the form on the $n$-th impulse. Suppose that $f(t)$ does not change its sign on a sampling interval $(t_n, t_{n+1})$, and a sampling period $T_n = t_{n+1} - t_n$ can be estimated

$$\kappa_0 T \leq T_n \leq T,$$

where $\kappa_0, T$ are positive constants; i.e. upper and lower bounds of $T_n$ exist.

Let vector $x(t)$ of dimension $m$ describe the system’s state at a moment $t$. Consider a functional equation (1) together with a linear differential equation with constant coefficients written in vector form

$$\frac{dx}{dt} = Ax + bf, \sigma = c^*x. \tag{3}$$

Here $A$ is a real constant $m \times m$ matrix, $b, c$ are real constant $m$-dimensional vectors, and $*$ denotes vector transpose. In technical applications equation (3) is commonly described by the rational complex-valued function $W(p)$ of a complex variable $p$: $W(p) = c^*(A - pI_m)^{-1}b$, with $I_m$ being the identity matrix of order $m$. It is called the transfer function from $-f$ to $\sigma$. Suppose that polynomials $\det(pI_m - A)$ and $W(p) \det(pI_m - A)$ have no common roots (such transfer functions are said to be nonsingular). Denote

$$\rho = \lim_{p \to 0} pW(p), \quad \kappa = \lim_{p \to \infty} pW(p), \quad \chi(p) = pW(p) - \kappa.$$

Let us consider the sequence $\{v_n\}$ of mean values of $f(t)$ at sampling intervals

$$v_n = \frac{1}{T_n} \int_{t_n}^{t_{n+1}} f(t) \, dt.$$

Require $v_n$ to be bounded for $n \geq 0$. For many kinds of modulation a nonlinear function $\phi(\sigma)$ can be constructed so that for any sufficiently large $n$, $n \geq 0$, a moment $\tilde{t}_n$ satisfying $t_n \leq \tilde{t}_n \leq t_{n+1}$ and

$$v_n = \phi(\sigma(\tilde{t}_n)) \tag{4}$$

exists. This function $\phi(\sigma)$ will be called an equivalent nonlinearity. Let us show that an equivalent nonlinearity exists for commonly used kinds of modulation.

1. Pulse-amplitude modulation (PAM). In this case $t_n = nT$,

$$f(t) = \begin{cases} a(\sigma(nT))/\tau, & nT < t \leq nT + \tau, \\ 0, & nT + \tau < t \leq (n + 1)T, \end{cases}$$

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with \( \tau, T \) positive constant, \( 0 < \tau < T \). Here \( a(\sigma) \) is a continuous bounded function, \( a(0) = 0 \). Obviously \( v_n = a(\sigma(nT))/T \), and (4) holds for \( \tilde{t}_n = nT \) and \( \phi(\sigma) = a(\sigma)/T \).

2. Pulse-frequency modulation of the first kind (PFM-1). Here \( t_{n+1} = t_n + \Phi(|\sigma(t_n)|) \),

\[
    f(t) = \begin{cases} 
        \lambda(\sigma(t_n))/\tau, & t_n < t \leq t_n + \tau, \\
        0, & t_n + \tau < t \leq t_{n+1}, 
    \end{cases}
\]

\( \lambda(\sigma) = 0 \) for \( |\sigma| \leq \Delta \) and \( \lambda(\sigma) = \text{sign}(\sigma) \) for \( |\sigma| > \Delta \). \( \tau, \Delta \) are positive constants. The function \( \Phi(\mu) \) is continuous, nonincreasing for \( \mu \geq 0 \) and \( \Phi(\mu) \rightarrow \Phi_\infty = \text{const} > 0 \) as \( \mu \rightarrow +\infty \), \( 0 < \tau < \Phi_\infty \). Apparently \( v_n = \lambda(\sigma(t_n))/\Phi(|\sigma(t_n)|) \) and (4) holds for \( \tilde{t}_n = t_n \) and \( \phi(\sigma) = \lambda(\sigma)/\Phi(|\sigma|) \).

3. Pulse-frequency modulation of the second kind (PFM-2). In this case \( t_{n+1} \) is the minimal root of the equation \( t_{n+1} = t_n + \Phi(|\sigma(t_n+1)|) \), the functions \( f, \lambda \) and \( \Phi \) are the same as those for PFM-1. Evidently \( v_n = \lambda(\sigma(t_n))/\Phi(|\sigma(t_n)|) \).

Since \( \lambda(\sigma(t_n)) \neq \lambda(\sigma(t_{n+1})) \), relation (4) does not hold directly. To solve this problem let us consider the sequence \( \hat{t}_n = t_n + \tau \) instead of \( t_n \). Then \( \hat{t}_{n+1} - \hat{t}_n = T_n = \Phi(|\sigma(t_{n+1})|) \) and \( \phi(\sigma) = \lambda(\sigma)/\Phi(|\sigma|) \) turns out to be an equivalent nonlinearity with \( \hat{t}_n = t_{n+1} \).

4. Pulse-width modulation of the first kind (PWM-1). Here \( t_n = nT \) \((T = \text{const} > 0)\),

\[
    f(t) = \begin{cases} 
        \text{sign}(\sigma(nT)), & nT < t \leq nT + \tau_n, \\
        0, & nT + \tau_n < t \leq (n+1)T, 
    \end{cases}
\]

\( \tau_n = F(|\sigma(nT)|) \). The function \( F(\mu) \) is continuous and nondecreasing when \( \mu \geq 0, F(0) = 0, F(\mu) \leq T \) for all \( \mu > 0 \). Evidently \( v_n = F(|\sigma(nT)|) \text{sign}(\sigma(nT))/T \) and (4) holds for \( \phi(\sigma) = F(|\sigma|) \text{sign}(\sigma)/T \) and \( \tilde{t}_n = nT \).

5. Pulse-width modulation of the second kind (PWM-2). In this case \( t_n = nT, \tau_n \) is the minimal nonnegative root, satisfying \( \tau_n \leq T \), of the equation \( \tau_n = F(|\sigma(nT + \tau_n)|) \), if it exists; otherwise, \( \tau_n = T \). The functions \( f(t) \) and \( F(\mu) \) are the same as those for PWM-1. It is obvious that \( v_n = F(|\sigma(nT + \tau_n)|) \text{sign}(\sigma(nT))/T \). The modulation law implies that \( \tau < F(|\sigma(nT + \tau)|) \) for \( \tau \in [0, \tau_n] \). Hence \( \sigma(nT + \tau) \neq 0 \) when \( \tau \in [0, \tau_n] \) and therefore \( \text{sign}(\sigma(nT)) = \text{sign}(\sigma(nT + \tau_n)) \). So (4) holds when \( \phi(\sigma) = F(|\sigma|) \text{sign}(\sigma)/T \) and \( \tilde{t}_n = nT + \tau_n \).

6. Integral pulse-width modulation (IPWM). In this case \( t_n = nT \),

\[
    f(t) = \begin{cases} 
        0, & nT < t \leq nT + \tau_n, \\
        \text{sign}(\mu_n(\tau_n)), & nT + \tau_n < t \leq (n+1)T, 
    \end{cases}
\]

\( \mu_n(\tau) = \int_0^\tau \sigma(nT+s) \, ds \).

Here \( \tau_n \) is the minimal positive root, belonging to the interval \((0, T] \), of the equation \(|\mu_n(\tau_n)| = \Delta \) (with \( \Delta \) a positive constant). If such a root
does not exist, then \( f(t) \equiv 0 \) for \( nT < t \leq (n + 1)T \). It is evident that \( v_n = (T - \tau_n) \text{sign} \left( \mu_n(\tau_n) \right) / T \), if the root \( \tau_n \) exists, and \( v_n = 0 \) otherwise. Apparently \( \mu_n(\tau_n) = \sigma(nT + \zeta_n)\tau_n \) for some middle point \( \zeta_n \), \( 0 \leq \zeta_n \leq \tau_n \). Hence, if the root \( \tau_n \) exists, then \( \tau_n = \Delta / |\sigma(nT + \zeta_n)| \) and

\[
v_n = \left[ 1 - \frac{\Delta}{T|\sigma(nT + \zeta_n)|} \right] \text{sign} \left( \sigma(nT + \zeta_n) \right).
\]

If the root \( \tau_n \) does not exist, then a number \( \eta_n \) such that \( nT < \eta_n < (n + 1)T \) and \( |\sigma(\eta_n)| < \Delta / T \) can be found. Therefore (4) follows by setting

\[
\phi(\sigma) = \begin{cases} 
(1 - \Delta/(T|\sigma|)) \text{sign}(\sigma), & |\sigma| \geq \Delta / T, \\
0, & |\sigma| < \Delta / T,
\end{cases}
\]

with either \( \tilde{t}_n = nT + \zeta_n \) or \( \tilde{t}_n = \eta_n \).

The equivalent nonlinearity can also be constructed for some more complicated types of modulation (combined, linear integral pulse-width, phase) [4].

3. The Main Results

Suppose that for any sufficiently large \( n, n \geq 0 \), a number \( \tilde{t}_n \) exists such that \( t_n \leq \tilde{t}_n \leq t_{n+1} \) and

\[
(\sigma(\tilde{t}_n) - \sigma_* v_n) v_n \geq 0,
\]

where \( \sigma_* \) is a nonnegative constant. If an equivalent nonlinearity \( \phi(\sigma) \) exists, then (5) is evidently guaranteed by the conditions: \( \phi(0) = 0 \) and

\[
0 \leq \frac{\phi(\sigma)}{\sigma} \leq \frac{1}{\sigma_*}
\]

for all \( \sigma \neq 0 \) (if \( \sigma_* = 0 \) then \( 1/\sigma_* = +\infty \)). The restriction (6) means that the graphic of the function \( y = \phi(\sigma) \) lies in the plane sector bounded by the straight lines \( y = 0 \) and \( y = \sigma/\sigma_* \). The Popov stability criterion is known for systems with nonlinearities satisfying (6) [5]. We will formulate the stability criterion for pulse-modulated systems which is converted to that of Popov as \( T \) tends to zero.

The first theorem is concerned with systems having stable linear part.

**Theorem 1.** Let matrix \( A \) be stable, i.e. all its eigenvalues lie in the open left half-plane, and let numbers \( \tau, \varepsilon, \theta, (\tau, \varepsilon \text{ positive}) \) exist such that the following conditions hold.

1. Provided \( \theta \neq 0 \), an equivalent nonlinearity \( \phi(\sigma) \) exists which satisfies (4), with \( \tilde{t}_n \) being the same as in (5), and the Lipschitz condition

\[
|\phi(\sigma_1) - \phi(\sigma_2)| \leq L|\sigma_1 - \sigma_2|
\]

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for all \( \sigma_1, \sigma_2 \). (Here \( L \) is a positive constant.)

2. The algebraic inequality

\[
\sigma_\ast + \theta \kappa > \tau + \varepsilon_2 + \varepsilon_1 \kappa^2
\]

(8)

takes place.

3. For all real numbers \( \omega, 0 \leq \omega \leq +\infty \), the frequency-domain inequality

\[
\sigma_\ast + ReW(i\omega) + \theta Re(i\omega W(i\omega)) - \tau - \varepsilon_2 - \varepsilon_1 \omega^2 |W(i\omega)|^2 - (T^2/(12\tau))|\chi(i\omega)|^2[4\varepsilon_1(\sigma_\ast - \tau - \varepsilon_2)\omega^2 + \theta^2 \omega^2 + 1] > 0
\]

(9)

holds. (For \( \omega = +\infty \) this inequality is considered as a limit relation.) Here the numbers \( \varepsilon_1, \varepsilon_2 \) are defined by formulas

\[
\varepsilon_1 = \frac{T^2}{\pi^2 \varepsilon}[1 + \frac{\pi}{2} |\theta| L|\kappa]|^2 + \frac{2|\theta|LT}{\pi}, \quad \varepsilon_2 = \varepsilon + T|\kappa|.
\]

Then for all the solutions of (1) and (3) the limit relations \( v_n \to 0 \) as \( n \to \infty \), \( \sigma(t) \to 0 \) as \( t \to +\infty \) take place.

It is of interest to observe that if \( T \to 0, \tau \to 0, \varepsilon \to 0 \) so that \( T^2/\varepsilon \to 0, T^2/\tau \to 0 \), then (8) and (9) are reduced to the Popov frequency-domain condition:

\[
\sigma_\ast + ReW(i\omega) + \theta Re(i\omega W(i\omega)) > 0 \quad \text{as} \quad 0 \leq \omega \leq +\infty.
\]

We proceed by treating the case of a system with a neutral linear part.

**Theorem 2.** Assume that matrix \( A \) has one zero eigenvalue while all then other eigenvalues have negative real parts. Let \( \rho > 0 \) and numbers \( \tau, \varepsilon, \theta (\tau, \varepsilon \) positive\) exist such that conditions 1–3 of Theorem 1 hold. (For \( \omega = 0 \) inequality (9) is considered as a limit relation.) Then for all the solutions of (1) and (3), the limit relations \( v_n \to 0 \) as \( n \to 0 \), \( \sigma(t) \to \sigma_{\infty} \) as \( t \to +\infty \) take place. Here \( \sigma_{\infty} \) is a finite number depending on initial conditions, if an equivalent nonlinearity exists, then \( \phi(\sigma_{\infty}) = 0 \).

The above formulated theorems can be strengthened for some important special cases.

**Theorem 3.** If for all sufficiently large \( n \), \( n \geq 0 \), either \( \tilde{t}_n = t_n \) or \( \tilde{t}_n = t_{n+1} \) holds, then Theorems 1 and 2 remain valid if we use the formulas

\[
\varepsilon_1 = \frac{T^2}{\pi^2 \varepsilon} + \frac{2|\theta|LT}{\pi}, \quad \varepsilon_2 = \varepsilon
\]

instead of formulas (10).

As shown previously, the condition \( \tilde{t}_n = t_n \) takes place for PAM, PFM-1 and PWM-1.
We shall define for positive numbers \( n \), such that \( t_n < \tilde{t}_n < t_{n+1} \), two values

\[
M_n = \frac{1}{t_n - t_n} \int_{t_n}^{\tilde{t}_n} |f(t)| \, dt, \quad N_n = \frac{1}{t_{n+1} - \tilde{t}_n} \int_{\tilde{t}_n}^{t_{n+1}} |f(t)| \, dt.
\]

**Theorem 4.** Let for all sufficiently large \( n, n \geq 0 \), satisfying \( t_n < \tilde{t}_n < t_{n+1} \), one of the following conditions holds:
1. \( \kappa > 0 \) and \( M_n \geq N_n \);
2. \( \kappa < 0 \) and \( M_n \leq N_n \).

Then Theorems 1 and 2 remain true for \( \varepsilon_2 = \varepsilon \).

Evidently \( M_n, N_n \) are mean values of a function \( |f(t)| \) on intervals \((t_n, \tilde{t}_n)\) and \((\tilde{t}_n, t_{n+1})\), respectively. The inequality \( M_n \geq N_n \) takes place when \( |f(t)| \) is nonincreasing on \((t_n, t_{n+1})\) or \( f(t) = 0 \) on \((\tilde{t}_n, t_{n+1})\). The inequality \( M_n \leq N_n \) holds when \( |f(t)| \) is nondecreasing on \((t_n, t_{n+1})\) or \( f(t) = 0 \) on \((t_n, \tilde{t}_n)\).

**Example.** Consider the equation of the first order \( dx/dt = -ax - kf, \sigma = x \), where \( k, a \) are positive parameters. In this case \( W(p) = k/(p + \alpha) \). We shall apply Theorem 1 with \( \theta = 0 \). Frequency inequality (9) can be reduced to two algebraic inequalities:

\[
\tau(\sigma - \tau - \varepsilon_2) - \varepsilon T^2 k^2 \alpha^2 (\sigma - \tau - \varepsilon_2)/3 > 0, \\
\tau(\sigma - \tau - \varepsilon_2) \alpha^2 + \tau k \alpha - T^2 k^2 \alpha^2/12 > 0.
\]

Selecting \( \tau = akT^2/(\pi \sqrt{3}) \), \( \varepsilon = (\sigma - Tk)/2 \), it is straightforward to check that the conditions of Theorem 1 hold if

\[
\sigma > (1 + 2/\pi)kT + 2akT^2/(\pi \sqrt{3}) .
\]

If either Theorem 3 or Theorem 4 is applicable, then by choosing \( \varepsilon = \sigma/2 \), the inequality

\[
\sigma > 2kT/\pi + 2akT^2/(\pi \sqrt{3})
\]

is obtained.

**4. Proofs of the Theorems**

Let us consider a piecewise constant function \( v(t) \): \( v(t) = v_n \) as \( t_n < t \leq t_{n+1} \). We shall change the variables in equation (3) with the help of Liénard-type transformation:

\[
u = \int_{t_0}^{t} (f(s) - v(s)) \, ds, \quad y = x - bu, \quad \tilde{\sigma} = \sigma + \kappa u.
\]

Taking into account that \( \kappa = -c^*b \), it leads to the equations

\[
\frac{dy}{dt} = Ay + Abu + bv, \quad \tilde{\sigma} = c^*y
\]  

for \( t \geq t_0 \). Let us define piecewise continuous functions \( \xi(t) = \tilde{\sigma}(t) - \tilde{\sigma}(\tilde{t}_n) \), \( \eta(t) = \tilde{\sigma}(t) - \sigma(\tilde{t}_n) \) when \( t_n < t \leq t_{n+1} \). Evidently \( \eta(t) = \xi(t) + \kappa u(\tilde{t}_n) \) for \( t_n < t \leq t_{n+1} \). First we prove three lemmas.
Lemma 1. The inequality

$$|u(t)| \leq T_n|v_n|$$  \hfill (12)

holds for any $t$, $t_n \leq t \leq t_{n+1}$, and the following relation holds

$$\int_{t_n}^{t_{n+1}} u(t)^2 \, dt \leq \frac{1}{3} v_n^2 T_n^3.$$

(13)

Proof of Lemma 1. Under the assumptions above the function $f(t)$ does not change its sign on $(t_n, t_{n+1})$. We shall consider the case when $f(t) \geq 0$ only; otherwise the proof is analogous. Let an interval $(\alpha, \beta)$, $t_n \leq \alpha < \beta \leq t_{n+1}$ exist such that $u(\alpha) = u(\beta) = 0$ and $u(t)$ does not change its sign on $(\alpha, \beta)$. Provided $f(t) \geq 0$, the following inequalities hold when $\alpha \leq t \leq \beta$:

$$u(t) = \int_{\alpha}^{t} (f(s) - v_n) \, ds \geq -v_n(t - \alpha),$$

(14)

$$u(t) = -\int_{t}^{\beta} (f(s) - v_n) \, ds \leq v_n(\beta - t).$$

(15)

If $u(t) \geq 0$ on $(\alpha, \beta)$, then (15) implies $0 \leq u(t) \leq v_n(\beta - t)$, so

$$\int_{\alpha}^{\beta} u(t)^2 \, dt \leq v_n^2 \int_{\alpha}^{\beta} (\beta - t)^2 \, dt = \frac{1}{3} v_n^2 (\beta - \alpha)^3.$$

If $u(t) \leq 0$ on $(\alpha, \beta)$, then we obtain from (14)

$$\int_{\alpha}^{\beta} u(t)^2 \, dt \leq v_n^2 \int_{\alpha}^{\beta} (t - \alpha)^2 \, dt = \frac{1}{3} v_n^2 (\beta - \alpha)^3.$$

Consider the set of points $t$ belonging to $(t_n, t_{n+1})$ for which $u(t) \neq 0$. Since $u(t)$ is a continuous function, this set is open and can be partitioned in not more than countable number of intervals $(\alpha_k, \beta_k)$, $k = 1, 2, \ldots$, which do not intersect each other. Then

$$\int_{t_n}^{t_{n+1}} u(t)^2 \, dt = \sum_{k=1}^{\infty} \int_{\alpha_k}^{\beta_k} u(t)^2 \, dt.$$

Obviously $u(\alpha_k) = u(\beta_k) = 0$ and $u(t)$ does not change its sign on $(\alpha_k, \beta_k)$. Therefore, as shown above,

$$\int_{\alpha_k}^{\beta_k} u(t)^2 \leq \frac{1}{3} v_n^2 (\beta_k - \alpha_k)^3.$$

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For an arbitrary integer $N$, $N \geq 1$, the estimate

$$\sum_{k=1}^{N} (\beta_k - \alpha_k)^3 \leq \left[ \sum_{k=1}^{N} (\beta_k - \alpha_k) \right]^3 \leq T_n^3$$

holds, whence

$$\sum_{k=1}^{\infty} (\beta_k - \alpha_k)^3 \leq T_n^3 .$$

Thus, inequality (13) is proved. Inequality (12) follows immediately from the estimate $-v_n(t - t_n) \leq u(t) \leq v_n(t_{n+1} - t)$ as $t_n \leq t \leq t_{n+1}$.

**Lemma 2.** The following inequality holds

$$\int_{t_n}^{t_{n+1}} \xi(t)^2 \, dt \leq \frac{4T_n^2}{\pi^2} \int_{t_n}^{t_{n+1}} \left[ \frac{d\sigma(t)}{dt} \right]^2 \, dt .$$

This inequality represents a version of the known Wirtinger inequality. It may be proved by slightly modified arguments given in [6].

Consider two real quadratic forms

$$F_1(u, v) = \tau(v^2 - 3u^2 / T^2) ,$$

$$F_2(y, u, v) = (c^*y - \sigma_*v)v + \theta v c^*(Ay + Abu + bv) +$$

$$+ \varepsilon_1 (c^*(Ay + Abu + bv))^2 + \varepsilon_2 v^2 .$$

**Lemma 3.** Let $n$ be a sufficiently large positive integer, and conditions of any theorem of those formulated above hold. Then for functions $y(t)$, $u(t)$, $v(t)$ satisfying (11) the following integral-quadratic relations are valid:

$$\int_{t_n}^{t_{n+1}} F_1(u(t), v(t)) \, dt \geq 0 , \quad (16)$$

$$\int_{t_n}^{t_{n+1}} F_2(y(t), u(t), v(t)) \, dt \geq \theta \int_{\sigma(t_n)}^{\sigma(t_{n+1})} \phi(\sigma) \, d\sigma . \quad (17)$$

**Proof of Lemma 3.** Inequality (16) evidently follows from Lemma 1. Denote $\Delta = 2T / \pi$. In order to prove (17), it suffices to show that there exists a nonnegative number $\theta_0$ such that

$$F_2(y, u, v) \geq \theta \phi(\sigma) \frac{d\sigma}{dt} + \theta_0 \left[ \Delta^2 \frac{d\sigma}{dt} - \xi^2 \right]$$

for sufficiently large $t$. Then Lemma 2 can be applied.
Rewrite (18) in the form of
\[(\tilde{\sigma} - \sigma_* v)v + \theta(v - \phi(\tilde{\sigma})) \frac{d\tilde{\sigma}}{dt} + (\varepsilon_1 - \theta_0 \Delta^2) \left[ \frac{d\tilde{\sigma}}{dt} \right]^2 + \varepsilon_2 v^2 + \theta_0 \xi^2 \geq 0. \quad (19)\]

Taking (5) and (7) into account, one sees that
\[(\tilde{\sigma} - \sigma_* v)v \geq \eta v, \quad |v - \phi(\tilde{\sigma})| \leq L |\eta|. \]

So (19) is ensured by the inequality
\[\eta v - |\theta| L |\eta| \left[ \frac{d\tilde{\sigma}}{dt} \right] + (\varepsilon_1 - \theta_0 \Delta^2) \left[ \frac{d\tilde{\sigma}}{dt} \right]^2 + \varepsilon_2 v^2 + \theta_0 \xi^2 \geq 0. \quad (20)\]

Let the conditions of Theorem 1 be fulfilled. In view of (12), the estimate
\[|\eta| \leq |\xi| + T |\kappa||v| \quad (21)\]
can be obtained. Hence for inequality (20) to hold, it suffices to ensure the inequality
\[(\varepsilon_1 - \theta_0 \Delta^2) \left[ \frac{d\tilde{\sigma}}{dt} \right]^2 + (\varepsilon_2 - T |\kappa|) v^2 + \theta_0 \xi^2 - |\xi||v| -
- |\theta| L |\xi| \left| \frac{d\tilde{\sigma}}{dt} \right| - |\kappa||\theta| L T |v| \left| \frac{d\tilde{\sigma}}{dt} \right| \geq 0. \]

Using \(\varepsilon = \varepsilon_2 - T |\kappa|\) we derive
\[\varepsilon v^2 - \left[ |\xi| + |\kappa||\theta| L T \left| \frac{d\tilde{\sigma}}{dt} \right| \right] |v| \geq - \frac{1}{4\varepsilon} \left[ |\kappa||\theta| L T \left| \frac{d\tilde{\sigma}}{dt} \right| + |\xi| \right]^2. \]

So (20) holds if the condition
\[\left[ \varepsilon_1 - \frac{1}{4\varepsilon} \theta^2 L^2 T^2 \kappa^2 - \theta_0 \Delta^2 \right] \left[ \frac{d\tilde{\sigma}}{dt} \right]^2 + \left[ \theta_0 - \frac{1}{4\varepsilon} \right] \xi^2 -
- \left[ |\theta| L + \frac{1}{2\varepsilon} |\theta| L T |\kappa| \right] \left| \frac{d\tilde{\sigma}}{dt} \right| |\xi| \geq 0\]
is fulfilled. Ensure the quadratic form, with arguments \(|\xi|, |d\tilde{\sigma}/dt|\), written in the left-hand side of this inequality to be positive semidefinite. Then \(\theta_0\) should be selected so that
\[\theta_0 \geq 1/(4\varepsilon), \quad \varepsilon_1 - (1/(4\varepsilon))\theta^2 L^2 T^2 \kappa^2 \geq \theta_0 \Delta^2, \quad (22)\]
4(\varepsilon_1 - \theta^2 L^2 T^2 \kappa^2/(4\varepsilon) - \theta_0 \Delta^2)(\theta_0 - 1/(4\varepsilon)) \geq \theta^2 L^2 (1 + T|\kappa|/(2\varepsilon))^2 . \quad (23)

Define \theta_0 by the formula \theta_0 = (4\varepsilon_1 - \theta^2 L^2 T^2 \kappa^2/\varepsilon + \Delta^2/\varepsilon)/(8\Delta^2). Substituting the right-hand side of this equality for \theta_0 in (23) provides

\[ (4\varepsilon_1 - \theta^2 L^2 T^2 \kappa^2/\varepsilon - \Delta^2/\varepsilon)^2 \geq 4\Delta^2 \theta^2 L^2 (2 + T|\kappa|/\varepsilon)^2 . \quad (24) \]

Inequalities (22) and (24) are satisfied if

\[ 4\varepsilon_1 \geq \theta^2 L^2 T^2 \kappa^2/\varepsilon + \Delta^2/\varepsilon + 2\Delta|\theta|L(2 + T|\kappa|/\varepsilon) , \]

which is equivalent to \varepsilon_1 \geq \Delta|\theta|L + (\Delta + |\theta|LT|\kappa|)^2/(4\varepsilon). This inequality is obviously valid when \varepsilon_1 is defined by (10). So relation (17) holds.

Under the conditions of Theorem 3 we obtain \( u(\tilde{t}_n) = 0 \) and \( \xi(t) = \eta(t) \). So we can use the relation \(|\xi| = |\eta|\) instead of (21) and the preceding arguments can be repeated as if \( \kappa = 0 \).

Let the assumptions of Theorem take place. It is easily seen that when \( t_n \leq t \leq t_{n+1} \) the relation

\[ T_n u(t) = (t_{n+1} - t) \int_{t_n}^{t} f(s) \, ds - (t - t_n) \int_{t}^{t_{n+1}} f(s) \, ds \quad (25) \]

holds. Under the assumptions imposed on \( f(t) \) we have \( v_n f(t) \geq 0 \) when \( t_n < t < t_{n+1} \). Put \( t = \tilde{t}_n \). Relation (25) and the conditions of Theorem 4 yield \( \kappa u(\tilde{t}_n) v_n \geq 0 \) when \( n \) is sufficiently large. Then \( \nu \eta \geq \nu \xi \) and the above proof is valid for \( \varepsilon_2 = \varepsilon \). The proof of Lemma 3 is completed.

**Proof of Theorem 1.** Consider the real quadratic form \( F(y,u,v) = F_1(u,v) + F_2(y,u,v) \), where \( F_1, F_2 \) are introduced earlier. Denote by \( \tilde{F}(\tilde{y}, \tilde{u}, \tilde{v}) \) the hermitian form (defined for complex \( \tilde{y}, \tilde{u}, \tilde{v} \)) obtained from \( F(y,u,v) \) by extending it to a hermitian form. For any real \( \omega \) we define the \( 2 \times 2 \) matrix function \( \Pi(i\omega) \) by the formula

\[ \tilde{F}(- (A - i\omega I_m)^{-1}(b\tilde{v} + A\tilde{u}), \tilde{u}, \tilde{v}) = - \left[ \begin{array}{c} \tilde{u} \\ \tilde{v} \end{array} \right]^* \Pi(i\omega) \left[ \begin{array}{c} \tilde{u} \\ \tilde{v} \end{array} \right] . \]

(Here * denotes hermitian conjugation.) By direct computations we obtain that

\[ \Pi(i\omega) = \begin{bmatrix} \Pi_{11}(i\omega) & \Pi_{12}(i\omega) \\ \Pi_{12}(i\omega) & \Pi_{22}(i\omega) \end{bmatrix} , \]

where

\[ \Pi_{11}(i\omega) = 3\tau/T^2 - \varepsilon_1 \omega^2 |\chi(i\omega)|^2 , \]

\[ \Pi_{22}(i\omega) = \sigma - \tau - \varepsilon_2 + ReW(i\omega) + \theta Re(i\omega W(i\omega)) - \varepsilon_1 \omega^2 |W(i\omega)|^2 , \]

\[ \Pi_{12}(i\omega) = \chi(-i\omega)(1/2 - \theta \omega/2 - \varepsilon_1 \omega^2 W(i\omega)) . \]

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Condition 2 of Theorem 1 means that \( \Pi_{22}(\infty) > 0 \). It is straightforward to show that condition 3 ensures the inequality \( \det \Pi(i\omega) > 0, \ 0 \leq \omega \leq +\infty \). By the Sylvester criterion and the continuity of \( \Pi(i\omega) \), the hermitian matrix \( \Pi(i\omega) \) is found to be positive definite for \( 0 \leq \omega \leq +\infty \). Then, by the Yakubovich-Kalman lemma in the nonsingular case (see Theorem 1.2.7 [7]), there exist a real symmetric matrix \( H \) and a positive number \( \delta_0 \) such that

\[
2y^*H(Ay + Abu + bv) + F(y, u, v) \leq -\delta_0 v^2
\]

for all real \( m \)-dimensional vectors \( y \) and real numbers \( u, v \). Define a Lyapunov function

\[
V(y) = y^*Hy + \theta \int_0^y \phi(\sigma) d\sigma.
\]

It follows from (26) and Lemma 3 that for any solution of (11) and for sufficiently large \( n, n \geq 0 \), the inequality

\[
V(y(t_{n+1})) - V(y(t_n)) \leq -\delta_0 T_n v_n^2
\]

holds. Under the above assumptions a sequence \( v_n \) is bounded and, consequently, so are functions \( v(t), u(t) \) for \( t \geq t_0 \). Therefore the stability of the matrix \( A \) ensured that all the solutions of (11) are bounded for \( t \geq t_0 \). Hence the function \( V(y(t)) \) is bounded as well, and (27) implies that

\[
\sum_{n=1}^{\infty} T_n v_n^2 < +\infty.
\]

Since, from (2), \( T_n \geq \kappa_0 T > 0 \), we conclude that \( v_n \rightarrow 0 \) as \( n \rightarrow \infty \). Then inequality (12) yields \( u(t) \rightarrow 0 \) as \( t \rightarrow +\infty \). As the matrix \( A \) is stable, we obtain \( y(t) \rightarrow 0, \sigma(t) \rightarrow 0 \) as \( t \rightarrow +\infty \). This completes the proof of Theorem 1.

The next lemma seems to be generally known. It will be useful in proving Theorem 2.

**Lemma 4.** Let \( \nu(t) \) be an absolute continuous function, \( g(t) \) be a piecewise continuous one, and

\[
(\nu(t) - g(t)) \frac{d\nu}{dt} \leq 0
\]

for all \( t \geq \tau_0 \). (At discontinuity points, (28) holds for both one-side limits.) Then the following statements are valid:

1. If \( g(t) \) is bounded for \( t \geq \tau_0 \), then so is \( \nu(t) \).
2. If \( g(t) \rightarrow 0 \) as \( t \rightarrow +\infty \), then \( \nu(t) \) has a finite limit as \( t \rightarrow +\infty \).
Proof of Lemma 4. Let $B$ be a number such that $g(t) \leq B$ for $t \geq \tau_0$ and $\nu(\tau_0) \leq B$. Show that $\nu(t) \leq B$ for $t \geq \tau_0$. Suppose not; then there exist numbers $\tau_1$, $\tau_2$ for which $\nu(\tau_1) = B$, $\nu(t) > B$ as $\tau_1 < t < \tau_2$. Then $\nu(t) - g(t) > 0$ for $\tau_1 < t < \tau_2$ and, consequently, (28) implies $d\nu/dt \leq 0$. So $\nu(t) \leq \nu(\tau_1) = B$ as $\tau_1 \leq t \leq \tau_2$. The contradiction proves $B$ to be an upper bound of $\nu(t)$. The lower bound of $\nu(t)$ may be obtained similarly.

Further, let $g(t) \to 0$ as $t \to +\infty$. If $d\nu/dt$ does not change its sign for sufficiently large $t$ then $\nu(t)$ becomes monotone, and hence it has a finite limit as $t \to +\infty$. Let $d\nu/dt$ take both positive and negative values when $t$ is large. Then $d\nu/dt - g(t) > 0$ for $T_1 < t < T_2$ and, consequently, (28) implies $dv/\nu(t)$ as $T_1 \to t \to T_2$. The contradiction proves $B$ to be an upper bound of $\nu(t)$.

Proof of Theorem 2. The transfer function $W(p)$ admits a representation

$$W(p) = W_1(p) + \rho/p,$$  \hspace{1cm} (29)

with the function $W_1(p)$ having all the poles in the open left half-plane. Then $ReW(i\omega) = ReW_1(i\omega)$ and it becomes obvious that the matrix function $\Pi(i\omega)$ defined in the proof of Theorem 1 depends continuously on $\omega$ when $0 \leq \omega \leq +\infty$. So, as shown in the proof of Theorem 1, $\Pi(i\omega)$ is positive definite for $0 \leq \omega \leq +\infty$. However, the Yakubovich-Kalman lemma in the nonsingular case is not applicable, since $W(p)$ has a zero pole. We use the singular case of this lemma (see Theorem 1.2.6 [7]). Let us consider the $2 \times 2$ matrix $\hat{\Pi}(i\omega)$ for which $\hat{\Pi}_{11} = \Pi_{11}$, $\hat{\Pi}_{12} = \Pi_{12}$, $\hat{\Pi}_{22} = \Pi_{22} - \delta_0$ with $\delta_0$ being a positive number. It is easily seen that $\hat{\Pi}(i\omega)$ is positive definite for $0 \leq \omega \leq +\infty$ when $\delta_0$ is sufficiently small. So there exist a matrix $H$ satisfying (26), and therefore (27) takes place. Further, by changing variables, the system (11) can be reduced to the system

$$\frac{dy_1}{dt} = A_1y_1 + A_1b_1u + b_1v, \quad \frac{dv}{dt} = v, \quad \sigma = c_1^*y_1 - \rho v - \kappa u.$$ \hspace{1cm} (30)

Here $y_1$ is a vector function of dimension $m - 1$, $A$ is a stable $m \times m$ matrix, $b_1$, $c_1$ are $m$-dimensional vectors, $\kappa = \rho - c_1^*b_1$, $c_1^*(A_1 - pI_{m-1})^{-1}b_1 = W_1(p)$ where $W_1(p)$ satisfies (29). Show that all the solutions of (30) are bounded when $t \geq t_0$. Since a sequence $v_n$ is bounded, we obtain that so are functions $u(t)$, $v(t)$ for $t \geq t_0$. The matrix $A_1$ being stable, we conclude that $y_1(t)$ is also bounded. Under assumption (5), we have $\sigma(t)v_n \geq 0$ for sufficiently large $n$. Since $\nu(t) = \nu(t) + (\tilde{t} - t)v_n$ for $t_n \leq t \leq t_{n+1}$ and $\rho > 0$, inequality (28) holds with

$$g(t) = (c_1^*y_1(\tilde{t}_n) - \kappa u(\tilde{t}_n))/\rho + v_n(\tilde{t}_n - t)$$  

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as $t_n < t < t_{n+1}$. By the first statement of Lemma 4, the function $v(t)$ is bounded for $t \geq t_0$. Thus, the solutions of (30) are bounded, and so are the solutions of (11). Similarly, as in the proof of Theorem 1, it can be shown that $v(t) \to 0$, $u(t) \to 0$ as $t \to +\infty$. Then $y_1(t) \to 0$ as $t \to +\infty$. Applying the statement of Lemma 4, we conclude that $v(t)$ has a finite limit as $t \to +\infty$ and, consequently, so does $\sigma(t)$.

The assertions of Theorem 3 and Theorem 4 follow immediately from Lemma 3 and the previous proofs.

References


