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ON SPECTRAL PROBLEM FOR A FUNCTIONAL DIFFERENTIAL EQUATION WITH MIXED CONTINUOUS AND DISCRETE MEASURE

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Abstract. The Sturm-Liouville boundary value problem for selfadjoint second order functional differential equation for the case of both continuous and discrete measures is considered. Discreteness of spectrum and a generalization of the Jacobi criterion for positivity of a quadratic functional are obtained.

Key Words. spectral problem, functional differential operator, positive solution, Green function

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1. The problem, notation, assumptions.

1.1. The problem. In [7] the eigenvalue problem for the self-adjoint (under some symmetry condition) singular equation

$$-(pu')' + p_1 u - \int_0^l (u(s) - u(x)) d_s r(x, s) = \lambda \rho u \tag{1.1}$$

is investigated (with Sturm-Liouville boundary condition at one end of the interval). From the point of view of mechanics, assume that there are point masses located at points $0 < x_1 < x_2 < \ldots < x_n < l$. For this reason, it is allowed discontinuity of the function pu' (in [7] the function p is measurable but pu' is equivalent to absolutely continuous one). We follow the scheme from [7], it was used also in [5, 3, 6].

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The main result are the theorems 8, 9. The theorem 8 on the equivalence of a number of important assertions for a particular but nonsymmetric case was obtained in [4]. The theorem 9 is important in the calculus of variations.

Let $I=[0,l]$ and $0=x_0<x_1<\dots<x_n<x_{n+1}=l$. The following

$$-(p(x)u')'+R(x)u-\int_I u(s)d_sr(x,s)=\rho(x)f(x),\ (x\neq x_i,\ i=1,\dots,n) \tag{1.2}$$

$$(pu')_{x_i-}-(pu')_{x_i+}+R_iu(x_i)-\sum_j d_{ij}u(x_j)=\rho_if(x_i)\quad(i=1,\dots,n) \tag{1.3}$$

boundary value problem with Sturm-Liouville boundary conditions

$$k_0u(0)-(pu')_{x=0}=0,\ k_1u(l)+(pu')_{x=l}=0. \tag{1.4}$$

can be considered as a certain mechanical model of a generalized string under the condition

$$R(x)\ge r(x,l)-r(x,0), \tag{1.5}$$

and $r(x,\cdot)$ does not decrease. Here condition (1.5) is omitted. As we will see below, a solution $u(x)$ of this equation is a continuous on $[0,l]$ function such that pu' is continuous in each segment $[x_i,x_{i+1}]$, $i=0,\dots,n$.

The equation (1.3) can be represented in form of equation (1.2) by means of the Dirac's Delta function. Instead, we use the notation

$$\frac{d}{d\mu}z=\begin{cases} z'(x), & \text{if } x\neq x_i\quad(i=1,\dots,n)\\ z(x+)-z(x-), & \text{otherwise.}\end{cases} \tag{1.6}$$

Then the system (1.2),(1.3) can be represented in the form[1]

$$-\frac{d}{d\mu}(pu')+R(x)u-\int_I u(s)d_sr(x,s)=\rho(x)f(x). \tag{1.7}$$

In this equation at $x=x_i$

$$R(x_i)=R_i,\ \int_I u(s)d_sr(x_i,s)=\sum_j d_{ij}u(x_j),\ \rho(x_i)=\rho_i.$$

[1] the μ can be considered as the sum of Lebesgue measure and a measure that is equal to unity at the points x_i

Note that in spite of the fact that the terms in the equation (1.7) are measurable functions, the values at points x_i play a special role (the measure of each of these points is equal to unity).

Along with this equation (1.7), we consider the equation

$$-\frac{d}{d\mu}(pu') + R(x)u - \int_I u(s)d_s r_0(x,s) = \rho(x)f(x). \tag{1.8}$$

It is short representation of the system

$$-(p(x)u')' + R(x)u - \int_I u(s)d_s r_0(x,s) = \rho(x)f(x),\ (x \neq x_i,\ i = 1,\dots,n) \tag{1.9}$$

$$(pu')_{x_i-} - (pu')_{x_i+} + R_i u(x_i) - \sum_j d^0_{ij}u(x_j) = \rho_i f(x_i), \quad (i = 1,\dots,n) \tag{1.10}$$

because we can define $r_0(x_i,s)$ so that

$$\int_I u(s)d_s r_0(x_i,s) = \sum_{j=1}^n d^0_{ij}u(x_j).$$

1.2. Assumptions. Assume that $k_0, k_1 \geq 0$ (the values $k_0 = +\infty$ and $k_1 = +\infty$ are admissible), $p(x)$ is measurable and positive almost everywhere on $[0,l]$ function, $1/p$ is Lebesgue integrable on $[0,l]$.

For each $y \in [0,l]$ the function $r_0(x,y)$ is integrable on $[0,l]$, for almost all x it does not decrease in y. It can be assumed that $r_0(x,0) = 0$. Suppose

$$R(x) \geq r_0(x,l),\ R_i \geq \sum_{j=1}^n d^0_{ij}. \tag{1.11}$$

Assume that

$$r(x,s) = r_0(x,s) + q(x,s),\ d_{ij} = d^0_{ij} + q_{ij} \quad (i,j = 1,\dots,n),$$

where $d^0_{ij} \geq 0$, $q_{ij} \geq 0$, $i,j = 1,\dots,n$, $q(x,\cdot)$ does not decrease on $[0,l]$, $q(\cdot,s)$ is measurable, $q(x,0) = 0$ and

$$\int_0^l q(x,l)^2\rho(x)\,dx < \infty. \tag{1.12}$$

The forms $\sum q_{ij}t_it_j$ and $\sum d^0_{ij}t_it_j$ are symmetric, that is, $q_{ij} = q_{ji}$, $d^0_{ij} = d^0_{ji}$, $i,j = 1,\dots,n$, and $q_{ij} \geq 0$.

1.3. Assumptions and bilinear forms. The measures ξ_0 and η are defined by the functions

$$\xi_0(x,y) = \int_0^x r_0(s,y)ds, \quad \eta(x,y) = \int_0^x q(s,y)\,ds.$$

They are symmetric, that is, $\xi_0(x,y) = \xi_0(y,x)$, and $\eta(x,y) = \eta(y,x)$ for $x, y \in [0,l]$. The initial object is the following two bilinear forms

$$[u,v] := k_0u(0)v(0) + k_1u(l)v(l) + \int_0^l pu'v'\,dx \tag{1.13}$$
$$+\int_0^l R(x)uv\,dx - \int_{I\times I} u(s)v(x)\,d\xi_0 + \sum_{i=1}^n R_iu(x_i)v(x_i) - \sum_{i,j=1}^n d_{ij}^0 u(x_j)v(x_i),$$

$$\langle u,v\rangle := [u,v] - Q(u,v), \tag{1.14}$$

where

$$Q(u,v) := \int_0^l u(s)v(x)\,d\eta - \sum_{i,j=1}^n q_{ij}u(x_j)v(x_i). \tag{1.15}$$

Note that the last term in (1.13) can be represented in the form

$$\sum_{i,j=1}^n q_{ij}u(x_j)v(x_i) = \int_{I\times I} u(s)v(x)\,d\xi_1,$$

where $\xi_1((x_i,x_j)) = q_{ij}$, that is, ξ_1 is a measure concentrated at the points (x_i,x_j).

1.4. Some notation. Let $L_2(I,\omega)$ be the Hilbert space of all measurable on I functions $f(x)$ with integrable square on I with nonnegative weight ω:

$$\int_I f(x)^2\omega(x)\,dx < \infty.$$

Here ω is parameter, we can write $L_2(I,\rho_1)$, $L_2(I,\rho_2)$, etc. The weight ω assumed to be nonnegative Lebesgue integrable. Since the values of a function $f(x)$ on the set $\omega_0 = \{x\colon \omega(x) = 0\}$ are not used, ω_0 can be added to sets of measure zero.

Let $\rho_1,\dots,\rho_n$ be positive numbers, and $\rho(x)$ be a fixed almost everywhere on $[0,l]$ nonnegative integrable function. We have to assume that

$$\int_0^l \rho(x)\,dx + \sum_{i=1}^n \rho_i > 0. \tag{1.16}$$

Let $L_2 = L_2(I, \rho, \rho_1, \dots, \rho_n)$ be the Hilbert space[2] of all functions from $L_2(I, \rho)$, but with scalar product

$$(f, g) = \int\limits_I f(x)g(x)\,\rho(x)\,dx + \sum_{i=1}^{n} f(x_i)g(x_i)\rho_i.$$

Note first that the values of the function at the points $x_1, \dots, x_n$ are essential, that is, these points do not belong to the sets of which we speak *for almost all*. Clearly, (f, g) can be represented in the form $\int_I f(x)g(x)d\mu$ of the integral with respect to the measure μ, which is the sum of Lebesgue measure and the measure concentrated in the points $x_1, \dots, x_n$. Note that the case $\rho(x) \equiv 0$ can also be considered. The space $L_2(I, 0, \rho_1, \dots, \rho_n)$ is a finite-dimensional Euclidean space.

Let W be the Hilbert space[3] of all absolutely continuous on $[0, l]$ functions u, satifying $[u, u] < \infty$. If $k_0 = +\infty$, assume that the function $u \in W$ satisfies the boundary condition $u(0) = 0$, and $u(l) = 0$ if $k_1 = +\infty$.

Define the operator T by the equality $Tu(x) = u(x)$, $x \in I$. The operator T acts from W to L_2 and it is continuous (lemma 7).

2. Results. First, consider the problem (1.8),(1.4). Applying to it the variational method (see the section 3), we obtain the solvability of the corresponding boundary value problem (1.8),(1.4) automatically. This problem is an equation $\mathcal{L}_0 u = f$ (where $\mathcal{L}_0 = (T^*)^{-1}$).

Using this boundary value problem, we then obtain the main theorem 8 on the equation $\mathcal{L}_0 u - Qu = f$, which represents the boundary value problem (1.7),(1.4). Note, that the operator Q appears because of the additional term in the form $\langle u, v\rangle$ (see (1.14)).

This scheme is presented in [7].

2.1. Positive form. Here it is the boundary value problem (1.8),(1.4). The following two theorems are presented in [9]. We quote them here for completeness. Note that the second part of the form (1.13) is positive because of conditions (1.11) and the identities

$$\frac{1}{2}\int_{I\times I} (u(s)-u(x))(v(s)-v(x))\,d\xi_0 = \int_0^l r_0(x, l)u(x)v(x)\,dx - \int_{I\times I} u(s)v(x)\,d\xi_0$$

[2] it follows from the definition
[3] it follows from the lemma 6

and

$$\frac{1}{2}\sum_{i,j=1}^{n}(u(x_j)-u(x_i))(v(x_j)-v(x_i))$$
$$=\sum_{i=1}^{n}\left(\sum_{j=1}^{n}d_{ij}^{0}\right)u(x_i)v(x_i)-\sum_{i,j=1}^{n}d_{ij}^{0}u(x_j)v(x_i)$$

This positivity allows form $[u,v]$ to be a scalar product in the space W (see lemma 6).

Theorem 1. *The boundary value problem* (1.8), (1.4) *is uniquely solvable in W for any $f\in L_2(I,\rho,\rho_1,\ldots,\rho_n)$.*

Proof. It follows from the lemmas 3 and 1. ∎

Theorem 2. *The eigenvalue problem*

$$-\frac{d}{d\mu}(pu')+R(x)u-\int_I u(s)d_s r_0(x,s)=\lambda\rho(x)u, \tag{2.1}$$

with boundary conditions (1.4) *has a system of eigenfunctions u_n corresponding to positive eigenvalues λ_n, $0<\lambda_0\le\lambda_1\le\ldots\le\cdots$, and* $\lim\lambda_n=+\infty$. *The system forms orthogonal basis in W.*

Proof. It follows from the corollary 1 and the lemma 2. ∎

Remark 1. *The minimal eigenvalue λ_0 satisfies the relation*

$$\lambda_0=\inf_{u\in W,u\neq 0}\frac{[u,u]}{\int_I u^2\rho\,dx+\sum_i u(x_i)^2\rho_i}.$$

This follows from the well known relation (see also [7])

$$\lambda_0=\inf_{u\in W,u\neq 0}\frac{[u,u]}{(Tu,Tu)}.$$

Remark 2. *In the case when $\rho(x)\equiv 0$, the space $L_2(I,0,\rho_1,\ldots,\rho_n)$ is finite dimensional euclidian space. In this case the system of eigenfunctions is finite.*

Remark 3. *Let $\rho(x)\equiv 0$ and $\xi_0\equiv 0$. In this case the equation* (1.9) *is $-(pu')'=0$. If $p=1$ its solution is a piecewise-linear function with discontinuity of the derivative at the points x_i. In general case the solution on each segment $[x_i,x_{i+1}]$ is represented by the integral $\int dx/p(x)$. It is absolutely continuous function and may have discontinuity of the derivative. The* *equation* (1.10) *becomes an linear algebraic system. It can be considered as a difference scheme. The eigenvalue problem* (2.1) *becomes an eigenvalue problem for the linear algebraic system.*

2.2. Green operator and Green function. The boundary value problem (1.9), (1.10), (1.4) has the short form (3.3). It has the unique solution $u = T^*f$. Here we see that the Green operator $G = T^*$ has mixed integral and sum representation. The proofs see in the section 3.4.

THEOREM 3. *The Green operator $G = T^*$ has the representation*

$$T^*f(x) = \int_I G(x,s)f(s)\rho(s)\,ds + \sum_{i=1}^{n} G(x,x_i)f(x_i)\rho_i. \tag{2.2}$$

For any $x \in I$ the function $g_x(s) = G(x,s)$ is an element in W. The Green function $G(x,s)$ is symmetric $G(x,s) = G(s,x)$.

REMARK 4. *The section $g_{x^*} = G(x^*,\cdot)$ satisfies the relation $u(x^*) = [u, g_{x^*}]$ for all $u \in W$ (this is the relation* (3.15)*).*

In [1] for equation without impulses it is showed that the section $G(\cdot,s)$ of the Green function is the solution of homogeneous problem, satisfying the condition $G'(s-,s) - G'(s+,s) = 1$. Here we have other proof for equation with impulses. Since $G(x,s)$ is symmetric $G(s,\cdot)$ can be considered instead of $G(\cdot,s)$.

THEOREM 4. *For any $x^* \in [0,l]$ the section $g_{x^*}(s) = G(x^*,s)$ of the Green function is the solution of the problem*

$$-\frac{d}{d\mu}(pu') + R(x)u - \int_I u(s)d_s r_0(x,s) = 0, \tag{2.3}$$

$$pu'|_{x^*-} - pu'|_{x^*+} = 1 \tag{2.4}$$

with boundary conditions (1.4).

THEOREM 5. *The Green function of the problem* (1.8),(1.4) *is positive in the square* $(0,l) \times (0,l)$.

2.3. General case of bilinear form. Here we consider the problem (1.7),(1.4).

THEOREM 6. *The problem* (1.7),(1.4) *is Fredholm one. In case of unique solvability it has integral representation with a kernel $G(x,s)$.*

Proof. It follows from the section 3, and from the lemmas 5, 10, 11. □

THEOREM 7. *The eigenvalue problem*

$$-\frac{d}{d\mu}(p(x)u') + R(x)u - \int_I u(s)d_s r(x,s) = \lambda\rho(x)u(x) \tag{2.5}$$

with boundary conditions (1.4) *has a system of eigenfunctions, asociated with eigenvalues* λ_n, $n = 0, 1, \ldots$. *It forms an orthogonal basis in* W.

The following theorem is a generalization of the well-known classical assertions. Note, that in this theorem the function $G(x, s)$ is the Green function of the problem (1.7), (1.4). It is different from the Green function of the problem (1.8), (1.4).

THEOREM 8. *The following statements are equivalent:*

1. *the quadratic functional* $\langle u, u\rangle$ *defined by* (1.14) *is positive definite,*
2. *the problem* (1.7),(1.4) *is uniquely resolvable, and its Green function is positive in* $I \times I$,
3. *the inequality*

$$-\frac{d}{d\mu}(p(x)u') + R(x)u - \int_I u(s)d_s r(x, s) \geq\not\equiv 0$$

has a positive in $(0, l)$ *solution,*

4. *the minimal eigenvalue of the problem* (2.5),(1.4) *is positive,*
5. *spectral radius of the operator* QT^* *is less than unit.*

In the case $k_0 = k_1 = \infty$, that is, if the boundary conditions have the form $u(0) = u(l) = 0$, there is one more equivalent statement. It is analog of the Jacobi criterion of positivity of the quadratic functional $\langle u, u\rangle$. Consider the truncated boundary value problem

(2.6)
$$-\frac{d}{d\mu}(p(x)u') + R(x)u - \int_\alpha^\beta u(s)d_s r(x, s) = 0,\ x \in [\alpha, \beta],\ u(\alpha) = u(\beta) = 0.$$

THEOREM 9. *The functional* $\langle u, u\rangle$ *is positive definite iff the problem* (2.6) *does not have nonzero solutions for any* $0 \leq \alpha < \beta \leq l$.

3. Variational method.

3.1. Equation in variational form.

The form (1.13) defines the equation

$$[u, v] = \int_0^l f(x)v(x)\rho(x)\,dx + \sum_{i=1}^n f(x_i)v(x_i)\rho_i, \quad v \in W. \tag{3.1}$$

In the equation (3.1) a function $v(x)$ ‘runs through’ the set W. This equation can be called *equation in variational form.* It can be represented in the short form

$$[u, v] = (f, Tv) \quad (\forall v \in W). \tag{3.2}$$

This equation may appear as the result of solving the problem of the minimum of the quadratic functional

$$\frac{1}{2}[u,u] - (f,Tu).$$

Existence and uniqueness of a solution of the equation (3.2) follows from the scheme in [7]. In that scheme it is supposed that $[u,v]$ is a scalar product in a Hilbert space W, and (f,g) is a scalar product in a Hilbert space H, $T\colon W \to H$ is a linear continuous operator, the image $T(W)$ is dense in H and $\dim\ker T = 0$. In our case $H = L_2 = L_2(I,\rho,\rho_1,\dots,\rho_n)$.

The operator T has the necessary properties (lemma 8). So,

Lemma 1. *Equation* (3.2) *is equivalent to an equation*

$$\mathcal{L}_0 u = f. \tag{3.3}$$

It has a unique solution in W for any $f \in L_2$.

Note that the solution of the equation (3.3) has the form $u = T^* f$. Thus, $\mathcal{L}_0 = (T^*)^{-1}$. The domain $D(\mathcal{L}_0) \subset W$ of the operator $\mathcal{L}_0$ is the image of the operator T^*, that is, $D(\mathcal{L}_0) = T^*(L_2)$. The adjoint operator T^* can be named *Green operator* $G = T^*$.

The eigenvalue problem in variational form

$$[u,v] = \lambda(Tu,Tv) \quad (\forall v \in W) \tag{3.4}$$

is equivalent to

$$\mathcal{L}_0 u = \lambda T u. \tag{3.5}$$

The operator T is compact (the lemma 10). Thus from the scheme in [7] it follows, that this problem has classic basic properties.

Lemma 2. *The eigenvalue problem* (3.5) *has a system of eigenfunctions u_n corresponding to positive eigenvalues λ_n, $0 < \lambda_0 \leq \lambda_1 \leq \ldots \leq \cdots$, and* $\lim \lambda_n = +\infty$. *The system forms orthogonal basis in W.*

3.2. Variational method in general case. For the bilinear form $\langle u,v\rangle = [u,v] - Q(u,v)$ (see (1.14)) a variational equation has the form

$$\langle u,v\rangle = (f,Tv), \ (\forall v \in W), \tag{3.6}$$

or $[u,v] - Q(u,v) = (f,Tv)$. Suppose $Q(u,v) = (Qu,Tv)$, where $Q\colon T^*(W) \to L_2$. Then this equation is equivalent to

$$\mathcal{L}_0 u - Qu = f. \tag{3.7}$$

The substitution $u = T^*z$ leads this equation to equation

$$z - QT^*z = f.$$

If the operator QT^* is compact, the equation (3.7) has Fredholm properties. Other details see in [7].

3.3. Euler equation. Boundary value problem.

LEMMA 3. *The boundary value problem* (1.8), (1.4) *represents the equation* (3.3).

Proof. Using the lemma 13 the equation (3.1) can be represented in the following complete form

$$\begin{aligned}(3.8)\quad k_0u(0)v(0) + k_1u(l)v(l) + \int_0^l pu'v'\,dx\\
+\int_0^l R(x)uv\,dx - \int_0^l v(x)\,dx\int_0^l u(s)d_sr_0(x,s)\\
+\sum_{i=1}^n R_iu(x_i)v(x_i) - \sum_{i=1}^n v(x_i)\sum_{j=1}^n d_{ij}^0u(x_j)\\
= \int_0^l f(x)v(x)\rho(x)\,dx + \sum_{i=1}^n f(x_i)v(x_i)\rho_i.\end{aligned}$$

Let first $v(x) = 0$ for $x \in [0,l] \setminus (x_i, x_{i+1}) \quad (i = 0,\dots,n)$. Then

$$\begin{aligned}\int_{x_i}^{x_{i+1}} pu'v'\,dx + \int_{x_i}^{x_{i+1}} R(x)uv\,dx - \int_{x_i}^{x_{i+1}} v(x)\,dx\int_0^l u(s)\,d_sr_0(x,s)\\
= \int_{x_i}^{x_{i+1}} f(x)v(x)\rho(x)\,dx.\end{aligned}$$

Let $h(x) = -R(x)u + \int_0^l u(s)\,d_sr_0(x,s) + f(x)\rho(x)$, $H(x) = \int_{x_i}^x h(s)\,ds$. Then

$$\int_{x_i}^{x_{i+1}} pu'v'\,dx = \int_{x_i}^{x_{i+1}} v(x)h(x)\,dx = -\int_{x_i}^{x_{i+1}} H(x)v'(x)\,dx.$$

From the lemma 9 it follows that $pu' + H$ is a constant. Thus, in (x_i, x_{i+1})

$$-(pu')' = h = -R(x)u + \int_0^l u(s)\,d_sr(x,s) + f(x)\rho(x).$$

From here it follows the first equation (1.9). Using (1.9) and

$$\int_0^l pu'v'\,dx = \sum_{k=0}^{n} \int_{x_k}^{x_{k+1}} pu'v'\,dx = \sum_{k=0}^{n} pu'v\Big|_{x_k}^{x_{k+1}} - \int_0^l (pu')'\,v\,dx \tag{3.9}$$

from (3.8) we obtain

$$\begin{aligned} k_0u(0)v(0) + k_1u(l)v(l) + \sum_{k=0}^{n} pu'v\Big|_{x_k}^{x_{k+1}} \\ + \sum_{i=1}^{n} R_iu(x_i)v(x_i) - \sum_{i=1}^{n} v(x_i)\sum_{j=1}^{n} d_{ij}^0u(x_j) = \sum_{i=1}^{n} f(x_i)v(x_i)\rho_i. \end{aligned} \tag{3.10}$$

Now, let in (3.10) $v(x_i) = 1$ for some $i \in \{1, 2, \dots, n\}$ but $v(x_k) = 0$ for all $k \neq i$. Then,

$$(pu'v)_{x_i-} - (pu'v)_{x_i+} + R_iu(x_i) - \sum_{j=1}^{n} d_{ij}^0u(x_j) = f(x_i)\rho_i$$

This is (1.10). Letting $v(0) = 1$ and $v(l) = 1$ obtain (1.4). ∎

Corollary 1. *The operator $\mathcal{L}_0$ defined in the lemma 1 and generated by the form* (1.13) *has representation*[4]

(3.11)
$$\mathcal{L}_0u(x) = \frac{1}{\rho}\left(-(pu')' + R(x)u - \int_I u(s)d_sr_0(x,s)\right) \quad (x \neq x_i,\ i = 1,\dots,n),$$
(3.12)
$$\mathcal{L}_0u(x_i) = \frac{1}{\rho_i}\left((pu')_{x_i-} - (pu')_{x_i+} + R_iu(x_i) - \sum_j d_{ij}^0u(x_j)\right) (i = 1,\dots,n).$$

The domain $D(\mathcal{L}_0)$ of the operator $\mathcal{L}_0$ consists of functions for which pu' is equivalent to an absolutely continuous on each $[x_i, x_{i+1}]$ function, satisfying the boundary conditions (1.4)*, and $\mathcal{L}_0u \in L_2$.*

Remark 5. *On the set $\{x\colon \rho(x) = 0\}$ the values of the operator $\mathcal{L}_0$ in the space $L_2(I, \rho, \rho_1, \dots, \rho_n)$ are not used. They can be attached to sets of measure zero. The equation $\mathcal{L}_0u = f$ is represented by equation* (1.8) *whose right-hand side is zero on this set.*

[4] see the remark 5

Lemma 4. *The form $Q(u,v)$ has the representation (Qu,Tv), where (see the remark 5)*

$$Qu(x) := \frac{1}{\rho(x)}\int_0^l u(s)d_s q(x,s),\ (x\neq x_i,\ i=1,\dots,n) \tag{3.13}$$

$$Qu(x_i) := \frac{1}{\rho_i}\sum_{j=1}^n q_{ij}u(x_j),\ (i=1,\dots,n) \tag{3.14}$$

Lemma 5. *The problem* (1.7),(1.4) *is represented by the equation* $\mathcal{L}_0 u - Qu = f$.

3.4. Proofs of theorems on Green's function.

Proof of the theorem 3. Let $u=T^*f$ and $x^*\in I$ is a fixed value. Then $u\mapsto u(x^*)$ is a bounded functional in W. Indeed, since $u(x^*)=u(0)+\int_0^{x^*}u'(s)\,ds$

$$\left(\int_0^{x^*}u'(s)\,ds\right)^2 \le \int_0^{x^*}\frac{ds}{p(s)}\int_0^{x^*}p(s)u'(s)^2\,ds \le [u,u]\int_0^{x^*}\frac{ds}{p(s)}.$$

Futher, $u(0)^2 = \dfrac{1}{k_0}k_0u(0)^2 \le \dfrac{1}{k_0}[u,u]$ (if $k_0=\infty$, $u(0)=0$). Thus this functional has the representation

$$u(x^*) = [u, g_{x^*}] \tag{3.15}$$

where $g_{x^*}\in W$. From here

$$u(x^*) = [T^*f, g_{x^*}] = (f, Tg_{x^*}) = \int_I g_{x^*}(s)f(s)\rho(s)\,ds + \sum_{i=1}^n g_{x^*}(x_i)f(x_i)\rho_i,$$

So, $G(x,s)=g_x(s)$.

The operator TT^* may be represented in the form (2.2) too, it is symmetric, and

$$\begin{aligned}(TT^*z_1,z_2) = &\int_I\left(\int_I G(x,s)z_1(s)\rho(s)\,ds\right)z_2(x)\rho(x)\,dx\\ &+\int_I\left(\sum_{i=1}^n G(x,x_i)z_1(x_i)\rho_i\right)z_2(x)\rho(x)\,dx\\ &+\sum_{i=1}^n\left(\int_I G(x_i,s)z_1(s)\rho(s)\,ds\right)z_2(x_i)\rho_i\\ &+\sum_{i=1}^n\left(\sum_{j=1}^n G(x_i,x_j)z_1(x_j)\rho_j\right)z_2(x_i)\rho_i\end{aligned}$$

Thus the function $G(x,s)\rho(s)\rho(x)$ is symmetric. ∎

Proof of the theorem 4. We do the same as it is in the proof of the lemma 3. The section $u = g_{x^*}$ satisfies the relation $v(x^*) = [v, g_{x^*}]$ for all $v \in W$ (remark 4). So,

$$\begin{aligned}(3.16)\quad v(x^*) &= k_0u(0)v(0) + k_1u(l)v(l) + \int_0^l pu'v'\,dx + \int_0^l R(x)uv\,dx \\ &- \int_I v(x)\,dx \int_I u(s)d_sr(x,s) + \sum_{i=1}^n R_iu(x_i)v(x_i) - \sum_{i,j=1}^n d_{ij}^0 u(x_i)v(x_j).\end{aligned}$$

Let $v(x) = 0$ in $I \setminus (x_i, x_{i+1})$, and $v(x^*) = 0$ (we can also include x^* in the set of points x_i). Since $0 = [v, g_{x^*}]$ it can be showed (as in the proof of the lemma 3) that

$$(3.17)\qquad -(pu')' + Ru - \int_I u(s)d_sr_0(x,s) = 0,\ x \in (x_i, x_{i+1}).$$

Here instead of (3.9) and because in the point x^* the function pu' may have discontinuity we will have

$$\begin{aligned}(3.18)\quad \int_0^l pu'v'\,dx &= \sum_{k=0}^n \int_{x_k}^{x_{k+1}} pu'v'\,dx \\ &= \sum_{k=0}^n pu'v\Big|_{x_k}^{x_{k+1}} + v(x^*)\left(pu'\big|_{x^*-} - pu'\big|_{x^*+}\right) - \int_0^l (pu')'\,v\,dx\end{aligned}$$

Substituting (3.18) and (3.17) in (3.16) obtain

$$\begin{aligned}v(x^*) = k_0u(0)v(0) + k_1u(l)v(l) + \sum_{k=0}^n pu'v\Big|_{x_k}^{x_{k+1}} + v(x^*)\left(pu'\big|_{x^*-} - pu'\big|_{x^*+}\right) \\ + R_iu(x_i)v(x_i) - \sum_{i,j=1}^n d_{ij}^0 u(x_i)v(x_j).\end{aligned}$$

From here in the same manner that in the proof of the lemma 3 we will have (2.3), (2.4), (1.4). ∎

4. Auxiliary propositions.

4.1. Properties of the basic space.

LEMMA 6. *The set W is a Hilbert space.*

Proof. We have to show completeness of the W. Suppose u_n is fundamental in W. From (1.13) it follows that u_n' is fundamental in $L_2(I,p)$, and $u_n(0) = 0$, or the sequence $u_n(0)$ is fundamental in R. So, $u_n' \to \varphi$ in $L_2(I,p)$, $u_n(0) \to c \in R$. The function $u(x) = c + \int_0^x \varphi(s)\,ds$ is limit of u_n in W. ∎

LEMMA 7. *$T(W) \subset L_2$, and the operator T is bounded.*

Proof. Since $u(x) = u(0) + \int_0^x u'(s)\,ds$, and $k_0 u(0)^2 \le [u,u]$, it is sufficient to consider the second term. Since

$$\|Tu\|^2 = \int_0^l u(x)^2\rho(x)\,dx + \sum_i u(x_i)^2\rho_i,$$

we have to check both terms:

$$\int_0^l \left(\int_0^x u'(s)ds\right)^2 \rho\,dx \le \int_0^l dx\,\rho(x)\int_0^x \frac{ds}{p(s)}\int_0^x p(s)u'(s)^2 ds$$
$$\le [u,u]\int_0^l dx\,\rho(x)\int_0^l \frac{ds}{p(s)}$$

and, since $u(x_i) = u(0) + \int_0^{x_i} u'(s)\,ds$,

$$\left(\int_0^{x_i} u'(s)\,ds\right)^2 \le \int_0^{x_i}\frac{ds}{p(s)}\int_0^{x_i} p(s)u'(s)^2 ds \le [u,u]\int_0^l \frac{ds}{p(s)}.$$

∎

LEMMA 8. *$T(W)$ is dense in L_2.*

Proof. If the closure $\overline{T(W)}$ in L_2 does not coincide with the whole space L_2, then there exists an $h \in L_2$ orthogonal to $T(W)$:

$$\int_0^l h(x)v(x)\rho(x)\,dx + \sum_{i=1}^n h(x_i)v(x_i)\rho_i = 0,\ \forall v \in W.$$

Let $v(x) = 0$ outside the interval $[x_i, x_{i+1}]$ and $\rho(x) \not\equiv 0$ in $[x_i, x_{i+1}]$. Then

$$\int_{x_i}^{x_{i+1}} h(x)v(x)\rho(x)\,dx = 0$$

for any such function. Let $H(x) = \int_{x_i}^x h(s)\rho(s)\,ds$, then

$$\int_0^l Hv'\,dx = \int_{x_i}^{x_{i+1}} Hv'\,dx = -\int_{x_i}^{x_{i+1}} h(x)v(x)\rho(x)\,dx = 0.$$

From the lemma 9 it follows that the function H is a constant, that is, $H(x) = 0$. So, $h = 0$. ∎

Lemma 9. *Suppose, p is a measurable and positive almost everywhere on $[a, b]$ function, the function $H(x)$ is measurable and such that $\int_a^b H(x)z(x)\,dx$ exists for any $z \in L_2([a, b], p)$. Suppose*

$$\int_a^b H(x)z(x)\,dx = 0 \tag{4.1}$$

for any $z \in L_2([a, b], p)$, such that $\int_a^b z(s)\,ds = 0$.

Then $H(x)$ is a constant.

Proof. Suppose first that $p(x)$ is bounded. Let $z(x) = H(x) - C$ where $C(b-a) = \int_a^b H(x)\,dx$. According to (4.1) $\int_a^b H(x)(H(x) - C)\,dx = 0$. Then

$$\int_a^b (H(x) - C)^2\,dx = 0,$$

whence $H(x) = C$.

In general case, for arbitrary $\varepsilon > 0$ it can be choosed $M > 0$ such that the measure of the set $e = \{x \in [a, b] \colon p(x) > M\}$ is less than ε. Similarly, we prove that $H(x) = C$ on $[a, b] \setminus e$. In this case let $z(x) = 0$ for $x \in e$. ∎

Lemma 10. *The operator T is compact.*

Proof. We will use the compactness criterium of Gelfand (see, for example [8, chapter 9]): for relative compactness of a set Ω in a Banach space E it is necessary and sufficient, that for any sequence f_n of linear bounded functionals converging to zero on any element the convergence be uniform on the set Ω. That is, for any sequence f_n

$$(\forall(x \in E))f_n(x) \to 0 \;\Rightarrow\; \sup_{x\in\Omega} |f_n(x)| \to 0.$$

Let $\Omega = \{Tu \colon \|u\|_W \le 1\}$. Suppose $f_n(z) \to 0$ for any $z \in L_2(I, \rho, \rho_1, \dots, \rho_n)$. We have

$$\begin{aligned} f_n(Tu) &= \int_I f_n(x)u(x)\rho(x)\,dx + \sum_i f_n(x_i)u(x_i)\rho_i \\ &= u(0)\int_I f_n(x)\rho(x)\,dx + \int_I dx\, f_n(x)\rho(x)\int_0^x u'(s)\,ds + \sum_i f_n(x_i)u(x_i)\rho_i. \end{aligned}$$

The first and the third terms are evidently tend to zero since $u(0)$ and $u(x_i)$ are bounded on the set $\|u\|_W \leq 1$ (see the proof of the lemma 7). Now consider the second term.

$$\left(\int_I dx\, f_n(x)\rho(x)\int_0^x u'(s)\,ds\right)^2 = \left(\int_I u'(s)\,ds\int_s^l f_n(x)\rho(x)\,dx\right)^2$$
$$\leq \int_I ds\, p(s)u'(s)^2\int_I \varphi_n(s)^2\,ds \leq [u,u]\int_I \varphi_n(s)^2\,ds,$$

where

$$\varphi_n(s) = \frac{1}{\sqrt{p(s)}}\int_s^l f_n(x)\rho(x)\,dx.$$

Note that $\varphi_n(s) = f_n(z_s)$, where $z_s(x) = 1/\sqrt{p(s)}$ if $x > s$ and $z_s(x) = 0$ if $x \leq s$ (and $z_s(x_i) = 0$ for $i = 1, \dots, n$). Thus, $\varphi_n(s) \to 0$ for any s. Since

$$\varphi_n(s)^2 \leq \frac{1}{p(s)}\int_s^l \rho(x)\,dx\int_s^l f_n(x)^2\rho(x)\,dx \leq \|f_n\|^2\,\frac{1}{p(s)}\int_s^l \rho(x)\,dx$$

by virtue of the Lebesgue theorem

$$\int_I \varphi_n(s)^2\,ds \to 0.$$

□

Lemma 11. *Under condition* (1.12) *the operator* QT^* *is compact.*

This lemma can be proved in a manner analogous to the method of Lemma 10.

4.2. Lemma about a generalization of the Fubini theorem. To deduce the Euler equation we need a transformation of an integral, that follows not from the Fubini theorem. Therefore we use the following assertion.

Lemma 12 ([2]). *Let* $(X,\mathcal{A})$ *and* $(Y,\mathcal{B})$ *be measurable spaces,* μ *be a measure on* $(X,\mathcal{A})$*,* $K\colon X\times\mathcal{B} \to [0,\infty]$ *be kernel (i.e. for* μ*–almost all* $x \in X$ $K(x,\cdot)$ *is a measure on* $(Y,\mathcal{B})$*,* $\forall B \in \mathcal{B}$ $K(\cdot,B)$ *is* μ*–measurable on* X*). Then*

1. *The function* ν *defined on* $\mathcal{A}\times\mathcal{B}$ *by the equality*

$$\nu(E) = \int_X K(x,E_x)\mu(dx), \quad E_x = \{y\colon (x,y) \in E\},$$

is measure.

2. if $f\colon X\times Y\to[-\infty,\infty]$ is ν–measurable on $X\times Y$, then

$$\int\limits_{X\times Y} f(x,y)d\nu=\int\limits_X\left(\int\limits_Y f(x,y)K(x,dy)\right)\mu(dx).$$

REMARK 6. *The function ν is Lebesgue extension from the set of rectangles*

$$\nu(A\times B)=\int_A K(x,B)\mu(dx),\quad A\in\mathcal{A},B\in\mathcal{B},$$

We now formulate the corresponding assertion, which follows from Lemma 12. In our case $X=Y=I=[0,l]$, the measure μ coincides with the Lebesgue measure.

LEMMA 13. *Let $f(x,y)$ be ξ-measurable function. Under assumptions in the section 1.2 it is valid the following relation:*

$$\int_{I\times I} f(x,s)\,d\xi=\int_I dx\int_I f(x,s)r(x,ds). \tag{4.2}$$

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