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DIFFERENTIAL EQUATIONS WITH RANDOM PARAMETERS: SCIENTIFIC SCHOOLS AND TECHNIQUE

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Abstract. Known and unknown research methods for stability of functional-differential systems are considered. More than hundred works in this field are reviewed. The author's own ideas for future scientific research are presented.

Key Words. Differential equations with random parameters, asymptotic stability of solution, systems of differential equations with delay, Markov's process.

AMS(MOS) subject classification. 49N10, 49K27, 34B27, 34K10

1. Introduce: Historical Review. The development of radiophysics, electrical engineering, automatic regulation and other sciences proved the necessity to take into account the different influence of undefined forces on dynamic systems. This condition forced creation of general theory of undefined function, in particular, the Theory of stochastic differential equations. The fundamental results in this area enclosed in works of Kolmogorov A. [52], Bernstein S. [13], Gikhman I. [29], Koroljuk V. [44], Dynkin J. [24], Skorohod A. [64], Katz I. [39], Krasovsky M. [40], Pugachov V. [60], Khasminsky R. [38], Stratonovich R. [67], Khinchin A. [33], Doob D. [18], Wiener I. [79], Feller V. [32], Ito K. [76] and others.
The considerable amount of works devoted the searching differential equations, which consist in view of parameters generalized undefined processes of white noise types. They can consider like equations, which taken by limiting crossing from the equations under impulsive influences, for example, the fractional effect in radio systems. For such equations there developed the special theory of stochastic differential equations in Gikhman I. and Skorohod A. works [29], which devoted to the stochastic differential equations, there considered the questions of existence and unity of stochastic differential

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equations solving; the connection of solving with processes of Markov's, the Kolmogorovs equations is got for transition probabilities of solving, studied the asymptotic behavior of solving with $t \longrightarrow \infty$ and characteristics of these equations solving.

In Khasminsky's work [38] researched the stochastic differential equations and other types of differential equations with undefined parameters. The method of auxiliary functions, i.e. the second Lyapunovs method, used in this case. The author gets the stable conditions of appropriate stochastic equations solving, studies characteristics of different stochastic equations solving. In this work, there is the huge bibliography, the attentive viewing in works of researching the linear systems with undefined parameters conducted.

The most part of works devoted to researching the parametrical influence of unexpected strength on the linear and nonlinear oscillating systems. Among them lets admit the works of Korenivsky D. [46,47], Stratonovich R. [67] and other authors. The most part of literature devoted to characteristics of the stationary process, investigation of stationary and periodic solutions of differential equations with random right-hand sides. We should point to the work of J. Doob [17], Rozanov Y. [62], Tsarkov E., Jasinsky V. [85] and others authors. Many works devoted to the application of the general theory of random functions, including the theory of differential equations with random functions to problems in mechanics, automatic control, radio technology, theoretical physics. Here we must point out the work of V.Zubov [83,84], I. Kazakov and V. Artemyev [9], B. Levin [59], A. Pervozvansky [60], A. Romanenko and G. Sergeyev [61], A. Sveshnikov [68], V.Tikhonov [70] and others. In the work [75] devoted to studying of the motion of gyroscopic systems under the influence of random forces. In the Korolyuk works [43] developed the theory of random evolutions. The general definition of Markov's random evolutions introduces in the work of [50]. The term 'random evolution' proposed by Lax. Theory of random evolutions obtained development from Papanicolaou [58], Pinsky [59], Kerts [41] works and others. In relation to the general definition of random evolutions there proposed the classification. Before all, we can the random evolution by type of switching processes: Markov's processes, renewal processes.

Then, we can distinguish a discontinuous evolution (if operators having jump), continuous (with absence of jump). Further, random evolution observed with continuous time, and discrete moments in times of recovery. In the theory of random evolutions, a prominent position occupies limit theorem in a series scheme.

Theorems of averaging for continuous Markov's random evolutions, for discontinuous, for fixed, diffusion approximation for continuous Markov's random evolutions studied in [85]. Limit theorems of averaging in scheme APC (asymptotic phase of consolidation) for Markov's random evolutions obtained in [50]. Diffusion approximation of evolutions and its scheme of phase averaging considered in [56]. For inhomogeneous Markov's processes evolutions there proposed algorithms for averaging and diffusion approximation [44]. An interesting aspect of the studying Markov evolutions as a multiplicative operator functions proposed by Pinsky M. [59]. The application of martingale methods for the study of random evolutions (the martingale property used for the characterization of Markov's processes by Struk A. and Varadan Sh.) allowed us to solve the problem in compactness of random evolutions in the series scheme and find the submission limit theorems evolutions in averaging and diffusion approximation. Also, the martingale approach allows estimating the rate of convergence in limit theorems. Martingale evolution properties and martingale multiplicative operator functionals properties considered in [41]. Research of independent and stationary evolutions using martingales given in [50]. The martingale approach to the study of stationary Markov's evolutions proposed in [56]. Limit theorems for Markov's processes martingale methods proved in [49]. The monograph of Skorokhod A. [64], where used martingale approach, dedicated to random evolution described systems of stochastic differential equations with fast Markov's switchings. To solve martingale problems for random processes in different spaces devoted many works. Martingale problem for discontinuous Markov's evolutions solved in [76], the multiplicative operator functionals of Wiener-in [79]. Solving the problem martingales for independent random evolutions dedicated the work [36]. The significant place among publications in random evolutions takes works about an application of these evolutions [26]. The most part of them connected with the name of Papanicolaou [58]. In the monograph [85], viewed the application of limit theorems for random evolutions to process inventories, transport, branching, diffusion, additive functionals and U statistical process, random motions on Lie groups and wave propagation in waveguides and bars, as well as fluctuations in the harmonic oscillator in a random environment [74]. There are stochastic models of systems that are not described by random evolution. Review of results and problems in the theory of Markov's random evolutions and multiplicative operator functionals of Markov's processes given in the papers [67]. In his paper [59], Pinsky considers multiplicative operator functionals and random evolution of Markov's processes and their applications.

The special role of Markov's processes in the modern theory of stochastic processes and its application recognized, and their research devoted to many studies on how different formulation of the problem, and to methods of their solution. These methods are often used in the presence of a sequence of stopping time, which breaks the temporal axis into intervals of the simple behavior of trajectories. The most studied processes of this type are jump processes, which the theory described in classical books of Gikhman Y. and Skorokhod A., Doob D. [17] and papers of Dynkin E. [23,24], Kolmogorov A. [45], W. Feller [27]. The attempt to build the foundations of the theory of Markov's processes whose behavior on the intervals between jumps are not trivial, made in the works of Watanabe S., Ikeda N. [76] and other authors. In these studies, the different procedures for constructing Markov's process parts observed. These constructions were fruitful in works of Skorokhod A. [64] and other authors noticed above who studied diffusion branching processes. The limitation, satisfied by the characteristics of these processes, have let to get some significant results, the presence of which associated not only with a complicated behavior of branching processes of diffusion between his jumps and not only with the variety of phenomena, due to branching. The Puasson"s complicated processes delayed at zero and common piecewise-linear processes investigated in detail in the works of Korolyuk V. [43], Kovalenko I. [48] and others. Important place among the considered processes takes so-called processes stockpiling of additive input with finite intensity jumps. Pioneering work in which we studied such processes, in all probability, is an article by J. Keylson and Memryn I. about the fractional effect, published in 1959. The attention of mathematicians to such processes have held in 1969 by Quran I. Since that time, the conditions of existence of one-dimensional measures of Levy's processes with simple wear and team studied, the type of infinitesimal operator was set. Zakusylo O. [81] in his work introduced the concept of the joint process stockpiling of additive input, set criterion for the existence of a stationary distribution of this process. The equations for the stationary distribution of the joint process stockpiling of additive input shown.These equations generalize the before obtained value and set different methods. The Laplace transform of time to achieve the lower level and the local behavior of processes with stable storage entrance researched. A number problems of queuing theory associated with the processes of life solved. The simplest multidimensional analogs processes stockpiling, discrete analogs processes with simple wear and its relationship with the theory tasks summation of independent random variables researched.

The works of Anisimov V. and Lebedev E. [7] focused on asymptotic methods. For networks of different types, there are conditions of existence of ergodic regime and multiplicativity of stationary distribution. On the base of the local classic and original recursive approaches that use limit theorems on the convergence of random processes to solutions of stochastic differential equations, there provided the averaging principle and diffusion approximation for a vector process number of requirements and latency in systems and networks such as Jackson type in transition mode and conditions big load.

2. Known and unknown methods: Formalizing goals and objectives of the study.

2.1. Differential equations with small parameters

Let's consider a system differential equations, written in vector-matrix form [63,64,75,85]

$$\dot{x(t)} = A(\varsigma(t))x(t), \tag{1}$$

where A continuous matrix function, $\varsigma(t)$ - Feller's and Markov's process to compact the equation with constant coefficients for the expectation of x solution

$$\dot{m(t)} = \sum_{k=0}^{\infty} \left(\epsilon^k A_k\right) m(t), m = Ex(t). \tag{2}$$

In [71-75] we study the exponential stability in mean square solution of the system (1). Trivial solution is called exponentially stable in mean square if there are positive constants N, γ , that at all $t \in R_l, x \in R^n, \varsigma \in Y$ for solutions to this initial inequality is used

$$m|x(t)|^2 \leq Ne^{-\gamma t}|x|^2$$

The necessary and enough condition for the stability of zero solution in the mean square, can be represente in the following form [85]:

Let there be continuous in y, symmetric matrix functions $\chi(y)$ and $\psi(y)$ such that the inequalities can be

$$C|x|^2 \leq x^T\chi(y)x \leq D|x|^2$$

$$C|x|^2 \leq x^T\psi(y)x \leq D|x|^2$$

and Lyapunov equation

$$\lim \frac{1}{t}[E_y \left(x^T(t)\psi(y(t))x(t) - \psi(y)\right)] = -\chi(y).$$

So then the zero solution of the system of equations is stable in the mean square.

In the Khrisanov's paper [38] obtained conditions of exponential stability of the first moment $E(t)$ of solution the system (2). Also, in this paper, we get an algorithm of stability analysis, which follows from the results of Tsarkov E.

In the above, studying the systems that undergo random effects. There used asymptotic and martingale methods. The Ordinary differential equations with random parameters studied in works [19], [20], [21], [22], [80-84]. Using ideas of averaging and costruct equations for the *stochastic operator or generator*. Solving many tasks to reduce system of equations to a system of lower order. Its simplifies the solution of the problem. The idea of reducing the number of equations called *the principle of reduction* [80]. Application of reduction principle for researching stability of differential equations' systems are given . In mechanics and control theory arises the many problem stability of differential equations. Its after the transformation takes the form

$$\dot{x(t)} = Ax(t) + \mu F(x, \mu), F(0, \mu) = 0 \tag{3}$$

where $F(x, \mu)$ - vector-function which, decomposes in a series of powers of a small parameter μ and differentiated enough times to x, μ. It was proved that the stability of solutions generating system of linear differential equations has the great role in representation of the solution in the form of expansions in series in the small parameter

$$\dot{x(t)} = Ax(t) \tag{4}$$

which depends on the location the roots of the characteristic equation in the complex plan

$$det(E\lambda - A) = 0 \tag{5}$$

If all the roots $\lambda_j(A)$ of equation (5) lie in the left half $-Re\lambda_j(A) \leq 0$, the zero solution of system (3) is asymptotic stable at sufficiently small values $|\mu|$. If at least one of the roots of the equation (5) has a positive real part, the solution of system (3) is unstable. Also there is a little studied case when $Re\lambda_j(A)$ and the characteristic parameters of part of the system of equations (4) is on the imaginary axis. This case is called *critical*.

In Lyapunov's paper [53] was chosen the method of a system constructing of differential equations of the form

$$\dot{x_1(t)} = A_1 x_1(t) + \mu F_1(x_1, \mu), dim x_1 < dim x \tag{6}$$

it has a smaller dimension than the dimension of the system of equations (3). This stability of a zero solution of the system (3) is equal to the stability of the zero solution of (6). All the eigenvalues of the matrix A_1 lie on the imaginary axis and coincide with pure imaginary numbers matrix A. The variables x_1 are called *critical variables.*
The study the stability of zero solution of (6) is easier than study the stability of the zero solution of system (3). Lyapunov's method worked in two case: if study stability of one zero root of the characteristic equation, i.e. $ImA = 0$, $dimx_1 = 1$ and in the case of a pair of pure imaginary roots, i.e. $\lambda_j(A_1) = i\omega$, $ImA = 0$, $dimx_2 = 2$. The transition from the system equations (3) to the system equations of smaller dimensions (6) has been named the construction. The idea of reducing a dimension of the system of equations (3) preserving the stability properties was called the Lyapunov's principle of reduction.
The basic idea is to construct a Lyapunov integral manifold of solutions of differential equation systems, which are used further. In many tasks of celestial mechanics and control theory there arises the problem of the zero solution's stability of differential equations, which after the transformation takes the form

$$\begin{aligned} \dot{x_1}(t) &= A_1x_1(t) + \mu F_1(x_1, x_2, \mu), Re\lambda_j(A_1) = 0 \\ \dot{x_2}(t) &= A_2x_2(t) + \mu F_2(x_1, x_2, \mu), Re\lambda_j(A_2) = 0, \end{aligned} \tag{7}$$

like $x_2 = \mu\Phi(x_1, \mu)$. Uncritical variables x_2 are served by critical variables x_1 on manifolds $x_2 = \mu\Phi(x_1, \mu)$. For this reason we search the vector function $\Phi(x_1, \mu)$ of the system of equations

$$\dot{x_2}(t) = \mu\frac{DF_1(x_1, \mu)}{Dx_1}\dot{x_1}(t).$$

For critical variables there is obtained the system of differential equations

$$\dot{x_1}(t) = A_1x_1(t) + \mu\Phi_1(x_1, \mu\Phi(x_1, \mu), \mu) \tag{8}$$

Lyapunov proved that the stability of the zero solution of equation (7) is equal to the stability of the zero solution of (8). Further reduction principle of Lyapunov was generalized in Banach spaces in the works Barbashyn E. [6], Valeev K. [71] and others.
Newton, Euler, Gauss, Lagrange, Laplace, Delaunay, Lindsmed , Poincare and many others researchers contributed to the asymptotic methods. These methods are used usually for solving problems of celestial mechanics. Its was described with canonical by differential equations.

Some versions of asymptotic integration were associated with the operation of averaging time t

$$\lim \frac{1}{T}\int_0^x (F(t,x(t)))dt = P(x)$$

Averaging is used in the works of B. Van der Pol, Mandelstam L., Papaleksi N. and others. In this works the construction of the averaged system has not been associated with a canonical system of differential equations. On the other hand, it was developed the method of normal forms in the works of Poincare A., Berkhof J., Zahel K., Bruno A., Markeev A. and many others. It turns out that the asymptotic method is a way of reduction of differential equations to normal form. The other, more simple and Algorithmical method is based on using the asymptotic method [18, 85]. Searching the stability of the system (8) with the zero solution is a very difficult task. As a rule, there is used the second Lyapunov method - the method of functions which was designed in works of Barbashin E. [6], Zubov V. [83,84], Valeev K. [71] and many others.

In Kyiv Maths School Bogoljubov M., Mitropolsky Y., Samoylenko A. [63] have done the great contribution to creation and development of the asymptotic method of numerical integration of differential equations. An averaging operation was generalized

$$Ef(t) = \lim p\int_0^t e^{-pt} f(t,x(t))dt,$$

where $\alpha \neq 0$, $Ee^{\alpha t} = 0$. In this case, the asymptotic method became available for normalization of equations (7), where the matrix of the linear approximation A has arbitrary eigenvalues, not only purely imaginary eigenvalues, i.e. for mixing and splitting critical variables without applying the principle of construction of Lyapunov.

In [53,71] there is considered a system of linear difference equations

$$\sum_{k=0}^{N} A_k x_{n+k} = 0, dim A_k = m \times m. \tag{9}$$

We look for integral manifold of solutions like system

$$x_{n+1} = Q_n x_n, (n = 0, 1, ..). \tag{10}$$

Matrices Q_n satisfy the matrix difference equation

$$A_0 + A_1Q_n + A_2Q_{n+1}Q_n + ... + A_nQ_{N-1+n}Q_{N-2+n}Q_n = 0.$$

If we denote by $z_j, j = 1, 2, ..., N$ roots of multiplicative equation

$$detA(z) = 0, A(z) = \sum_{k=0}^{n} z^k A_k,$$

then C_j - eigenvectors of matrix beam we can find out from the equations

$$A(z_j)C_j = 0, j = 1, 2, ..., N$$

Let $z_1, z_2, ..., z_m$ the largest group of absolute values of solutions. If those vectors $C_1, C_2, ..., C_n$ are linearly independent, then almost any solution of matrix difference equation (10) has a limit $Q = \lim Q_n$ and the eigenvalues of the matrix Q coincide with numbers $z_1, z_2, ..., z_m$. The system of difference equations (10) is an integral manifold of solutions

$$x_{n+1} = Qx_n \tag{11}$$

It's proved that the system of difference equations (9) throws to the equitation system (11). To have the zero solution of the system (9) stable (asymptotic stable, unstable) it is necessary and enough to have the zero solution of the system (11) stable (asymptotic stable, unstable).
In the works [20-21] there is considered systems of nonlinear differential equations

$$x_{n+1} = x_n + hF_0(x_n) + hF_1(x_{n-1}) + ... + hF_N(x_{n-N}) \tag{12}$$

where $F_k(x)$ - continuous and differentiable vector- function. It has been proven that the system of difference equations (11) reduces to the system of difference equations of the form

$$x_{n+1} = x_n + h\Phi(x_n, h). \tag{13}$$

Any solution of system (12) at sufficiently small values $|h|$ directs at n to one of the solutions of equations (13).
In particular, it means that any numerical method for integrating the system of differential equations

$$\dot{x(t)} = F(t, x(t))$$

for example, the Adams method

$$x_{n+1} = x_n + h \sum_{k=0}^{N} a_k F(t_{n-k}, x_{n-k}), t_n = nh$$

reduces to the Euler method

$$x_{n+1} = x_n + h\Phi(t_n, x_n, h)$$

In the works [16], [19] we consider the system of equations

$$\begin{aligned} \dot{x(t)} &= Ax(t) + \Phi(x, y), \Phi(0, 0) = 0, Re\lambda_j(A) = 0 \\ \dot{y(t)} &= By(t) + \Psi(x, y), \Psi(0, 0) = 0, Re\lambda_j(B) < 0 \end{aligned}$$

where $\Phi(x, y)$, $\Psi(x, y)$ - differentiable vector-functions and there is proposed the following algorithm analysis:

I. We should search integral manifold of solutions like $y = G(x), G(0) = 0$. Then take the set of coordinate functions $y = G_k(x), G_k(0) = 0, (k = 1, 2, ..., N)$ and

$$G(x, \alpha) = \sum_{k=1}^{N} \alpha_k G_k.$$

Vector $G(x, \alpha)$ satisfies the system of differential equations

$$BG(x, \alpha) + \psi(x, G(x, \alpha)) = \frac{DG(x, \alpha)}{Dx}(Ax + \Phi(x, G(x, \alpha))). \tag{14}$$

Numerically we search coefficients α_k in the expansion $G(x, \alpha)$ by the Galerkin method or the method of collocation. For this reason we take the terms x_j and find coefficients of the systems of equations

$$BG(x_j, \alpha) + \psi(x, G(x_j, \alpha)) = \frac{DG(x_j, \alpha)}{Dx}(Ax_j + \Psi(x_j, G(x_j, \alpha))). \tag{15}$$

II. Numerically the system of equations (14) is integrated with randomly given initial data x_{0j},y_{0j} at $t = 0$. At sufficiently large $T > 0$ we find values $x_j(T)$ and using the interpolation formula, we find $y = G(x)$. Stability of the zero solution of (15) approximately corresponds to the stability of zero solution of system

$$\dot{x(t)} = Ax + \Phi(x, G(x)).$$

III. Synthesis of optimal control is made by using the principle of reduction.

Let the optimal control $u = U(x)$ in a system of differential equations be searched

$$\dot{x(t)} = F(x, U(x)), F(0,0) = 0, dimx = m \tag{16}$$

that minimizes the functional integral

$$J = \int_0 w(x(t), U(t))dt$$

and let the system of implicit equations

$$\psi \frac{DF(x,U)}{DU} + \frac{Dw(x,U)}{DU} = 0 \tag{17}$$

will be resolved relatively to $u = \Phi(x, \psi)$.
The necessary optimality conditions lead to a system of differential equations

$$\begin{aligned} \dot{x(t)} &= F(x, \Phi), \\ \dot{\psi(t)} &= -\psi \frac{DF(x,\Phi)}{Dx} - \frac{Dw(x,\Phi)}{Dx}. \end{aligned} \tag{18}$$

Of all the solutions of the manifold of the system (18), which has dimension $2m$, only integral manifold of dimension m goes to zero the solution at $x = 0, \psi = 0$. To find this variety several points $x_k(0), \psi_k(0)$, are asked in the vicinity of zero solution and the system of equations (18) should be integrated back. At every step of integrating expression is numerically found from the system of equations (17). Let the found points $x_k(-T_k), \psi_k(-T_k), u_k(-T_k)$. The synthesis of optimal control is lead down to recovery the vector-function which is found by the points $x_k(-T_k)$, $u_k(-T_k)$.

3. Differential equations with delay. In [20], [39] we look at the system of differential equations with delay argument

$$\frac{dx(t)}{dt} = F(t, x(t), x(t-\tau)), \tag{19}$$

which has a zero solution, i.e. $F(t, 0, 0) = 0$
There is proposed a method for studying the stability of the zero solution with sufficiently small delay argument. The method is based on the construction of the system of equations (19) to a system of equations without delay argument.

The equations with delay argument were considered in Bellman [7], Elzgoltz and many others. In [3] proved that any solution of the differential equation with delay

$$\frac{dx(t)}{dt} = F(t, x(t), x(t-h)) \tag{20}$$

is possible to represent with some special solutions of equation (20) in a rather broad sense, if the delay does not exceed certain limits.
This idea was geeneralized to a system of differential equations with delay argument. That is rejected in the Valeev papers: he investigated the stability of the zero solution of functional differential equations

$$\begin{aligned} \dot{x(t)} &= Ax(t) + \mu\Phi(t, x(t+\theta_1), y(t+\theta_2)), \\ \dot{y(t)} &= Bx(t) + \mu\Psi(t, x(t+\theta_3), y(t+\theta_4)), \end{aligned} \tag{21}$$

where Φ, Ψ - functionals for vector functions , the vector functions $x(t+\theta_k)$, $y(t+\theta_k)$ satisfies the Lipschitz conditions relative to x, y . The system of equations (21) at fairly general assumptions can be reduced to a system of ordinary differential equations of the form

$$\dot{x(t)} = Ax(t) + \mu S(t, x(t), \mu), \tag{22}$$

such way that the stability of the zero solution of (21) was equivalent to the stability of the zero solution of (22). The results show how important it would be to apply the principle of reduction to Rotation theory of random processes.

3.1. System with the quadratic right-hand side of the plane. This part is processed according to [3]. More good results of the assessment of convergence of systems with quadratic right-hand side can be obtained by considering a a system of the next form

$$\begin{aligned} \dot{x_1(t)} &= a_{11}x_2(t) + a_{12}x_2(t) + b_{11}^1 x_1^2(t) + 2b_{12}^1 x_1 x_2 + b_{22}^1 x_2^2(t), \\ \dot{x_2(t)} &= a_{21}x_1(t) + a_{22}x_2(t) + b_{11}^2 x_1^2(t) + 2b_{12}^2 x_1 x_2 + b_{22}^2 x_2^2(t). \end{aligned}$$

By using notations

$$A := \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B_1 := \begin{bmatrix} b_{11}^1 & b_{12}^1 \\ b_{21}^1 & b_{22}^1 \end{bmatrix}, \quad B_2 := \begin{bmatrix} b_{11}^2 & b_{12}^2 \\ b_{21}^2 & b_{22}^2 \end{bmatrix},$$

the last system can be rewritten in the following vector-matrix form

$$\dot{x(t)} = Ax(t) + X^T(t)Bx(t), \tag{23}$$

Then full derivative of the Lyapunov function in the system (2) has the form

$$\frac{dV(x(t))}{dt} \leq -\left[\lambda_{min}(C) - 2 \mid H \mid\mid B \mid\mid x(t) \mid\right] \mid x(t) \mid^2, \tag{24}$$

where

$$\mid H \mid = \lambda_{max}(H) = \frac{1}{2}\left\{h_{11} + h_{22} + \sqrt{(h_{11} - h_{22}) + 4h_{12}^2}\right\},$$

$$\lambda_{min}(C) = \frac{1}{2}\left\{c_{11} + c_{22} + \sqrt{(c_{11} - c_{22}) + 4c_{12}^2}\right\},$$

$$\mid B \mid = \left\{\lambda_{max}(B^T B)\right\}.$$

Guaranteed stability region will be the interior of the ellipse

$$h_{11}x^2 + 2h_{12}xy + h_{22}y^2 \leq r_0^2.$$

3.2. System with a dedicated linear part. For the system with the linear part can be written as [3]

$$\dot{x_i}(t) \;=\; \left[a_i - \sum b_{ij}x_j(t)\right] x_i(t),$$

Here A is the square diagonal matrix with constant coefficients $A = \{a_{ii}\}$, $B = \{B_1, B_2, ..., B_n\}^T$ is the rectangular matrix consisting of a symmetric square matrices B_i, in which to place i column is the vector $b_i^T = (b_{i1}, b_{i2}, ..., b_{in})$, $X^T = \{X_1(t), X_2(t), ..., X_n(t)\}$ is rectangular $n \times n^2$ matrix, which consisting of matrix $X_i(t)$, in which the i rows are vectors $x(t)$ other elements zero.

Let's suppose, that $\det B \neq 0$ and

$$B := \begin{bmatrix} b_{11} & b_{21} & ... & b_{n1} \\ b_{12} & b_{22} & ... & b_{n2} \\ . & . & ... & . \\ b_{1n} & b_{2n} & ... & b_{nn}, \end{bmatrix},$$

Then, as a rule, the interest for the searching is a singular point $x_0^T = (x_1^0, x_2^0, ..., x_n^0)$ that is the solution of algebraic equations

$$Bx_0 = a, \quad a^T = (a_1, a_2, ..., a_n)$$

and located in the first quadrant, i.e., $x_i^0 > 0$.

After replacing $x(t) = y(t) + x_0$ and transformation, we get the system of equations of perturbations

$$y(\dot{t}) = \bar{A}y(t) + y(t)^T By(t). \tag{25}$$

Let's suppose that the matrix defined in (25) is asymptotically stable, i.e., $Re\lambda_i(\bar{A}) < 0$. Then the singular point $x_0^T = (x_1^0, x_2^0, ..., x_n^0)$ is asymptotically stable and the region of it's stability can be assessed using a quadratic Lyapunov function $V(y) = y^T Hy$, which is symmetric positive definite matrix is a solution of the Lyapunov's equation [5]

$$\bar{A}^T H + H\bar{A} = C.$$

Here C is an arbitrary, symmetric, positive definite matrix.

In the case of asymptotic stability of the matrix $\bar{A}$, the guaranteed stability of the equilibrium area of the singular point is inside the ellipse $y^T Hy = r^2$ is inside the sphere $\mid y \mid= R$. Denoting

$$G_0 = \left\{ y \in R^n : \mid y \mid < \frac{\lambda_{min}(C)}{2 \mid H \mid\mid B \mid} \right\},$$

we find that the area "guarantee" stability has the form [4]

$$G_{r_0} = max \left\{ G_r : G_r \subset G_0 \right\},$$
$$G_r = \left\{ y \in R^n : x^T Hx < r^2 \right\}.$$

and ellipse $y^T Hy = r^2$ should be placed inside a sphere of radius

$$R = \frac{\lambda_{min}(C)}{2 \mid H \mid\mid B \mid}$$

and "stretched" $r \longrightarrow \infty$ as long as the ellipse touches the sphere.

We obtain estimates convergence of solution initial state lay in 'guaranty' stability regions.

3.3. Model with delay. The models of the economy and the environment inherent lag factor, defined "the time of puberty," or "the time of the decision". Are thus more appropriate mathematical model describing the system of functional differential equations with delay [2],[4],[8, 10,[12],[14],[15],[16],[25], [26],[28], [30],[31],[34],[35],[37],[54],[57], [65,69,75,76]. The first mathematical models described by differential equations with constant delay were the equations Hutchison and Voltaire [1,3]. Competition generally occurs between the new population and the population born with retardation: in this case, population dynamics is determined by the equation Hutchison (1948), which has the form of a the differential equation with delay

$$\frac{dx(t)}{dt} = ax(t)\left(1 - \frac{x(t-\tau)}{k}\right). \tag{26}$$

The delay is due to the finite time required to achieve the "time of puberty."

Dynamical system described by equation (26) has two equilibrium $x(t) \equiv 0$ and $x(t) \equiv k$. It is easy to see that the linear approximation is given by the equation at the point $x = 0$ and points to the instability of the zero equilibrium.

Estimate the stability region in the phase space of the equilibrium position $x(t) = k$ of the original non-linear system (26). After the transformation $x(t) = y(t) + k$ point $x(t) = k$ to the origin, we obtain the equation

$$\frac{dy(t)}{dt} = -a\left[y(t) + k\right] y(t-\tau).$$

We use a quadratic Lyapunov function $V(y) = \frac{1}{2}y^2$. Thus, when

$$a\left[1 - \frac{1}{k} \mid y(t) \mid\right] > \left(\frac{a}{k}\right)^2 [\mid y(t) \mid]^2 \tau$$

the total derivative of the Lyapunov function is negative definite. and the stability conditions are determined by the inequalities

$$\mid y(t) \mid < k,$$

$$\tau < \frac{k[k- \mid y(t) \mid]}{a[\mid y(t) \mid +k]^2}.$$

In the universal vector-matrix form, the quadratic model with delay is written as

$$\dot{x(t)} = \left[Ax(t) + x^T(t-\tau)\right] Bx(t),$$

Making the substitution in $x(t) = y(t) + k$, we obtain the system of equations of the perturbation which, after transformation, takes the form

$$\dot{y(t)} = \bar{A}y(t) + y^T(t-\tau)By(t). \tag{27}$$

$$\bar{A} := \begin{bmatrix} b_{11}x_1^0 & b_{21}x_1^0 & ... & b_{2n}x_1^0 \\ b_{21}x_2^0 & b_{22}x_2^0 & ... & b_{2n}x_2^0 \\ . & . & ... & . \\ b_{1n}x_n^0 & b_{1n}x_n^0 & ... & b_{nn}x_n^0 \end{bmatrix}.$$

We study the stability of the zero equilibrium state of the system (27) using the method of Lyapunov functions quadratic form $V(y) = y^T Hy$. In assessing the total derivative is used Razumihin's condition [8]. For the function $V(y)$ it has the form

$$| y(t-\tau) | \leq \sqrt{\varphi(H)} \mid y(t) \mid .$$

The set of points $y \in R^n$ that are within the level surfaces $V(y) = \alpha$ of a Lyapunov functions $V(y) = y^T Hy$ by V^α, and its boundary by ∂V^α , i.e.

$$V^\alpha = \{y \in R^n : V(y) < \alpha\}$$

Theorem 1. *Let the solutions $y(t)$ of (27) is performed $y(T) \in \partial V^\alpha$ at the time $t = T > \tau$,and at $-\tau \leq t < T$ will $y(t) \in V^\propto$. Then holds the inequality*

$$|y(T) - y(T-\tau)| \leq \left[\bar{A} + B\sqrt{\varphi(H)}y(T)\right] \sqrt{\varphi(H)}y(T)\tau.$$

Theorem 2. *Let the $-\tau \leq t \leq 0$ initial conditions $\varphi(t)$ for the solutions $y(t)$ is implemented $\varphi(t) < \delta$. Then this solutions $y(t)$ on interval $0 \leq t \leq \tau$ be implemented*

$$|y(t)| \leq \delta \exp\left[\bar{A} + B\delta\right]\tau.$$

Theorem 3. *Let the matrix $\bar{A}$ is asymptotic stability: $Re\lambda_i(\bar{A}) < 0$. Then as $\tau < \tau_0$, where*

$$\tau_0 = \frac{\lambda_{min}(C)}{2HB\sqrt{\varphi(H)}},$$

equilibrium is asymptotic stable.

4. Ordinary Differential equation with Markov coefficients.

4.1. Stochastic operator (generator) and its properties

Let on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ defined two random variables $x = x(\omega)$ and $y = y(\omega)$, and the density distribution $f_2(y)$ of a random variable y determines by the density distribution $f_1(x)$ of a random variable x by the operator L, namely:

$$f_2(y) = Lf_1(x)$$

The operator L is called **stochastic operator** [17],[25]. In the general case, we introduce a set S of functions $f(x) = f(x_1, ..., x_m)$ such that

$$f(x) \gneqq 0,$$

$$\int_{E_m} f(x)dx = 1$$

and consider the operator L, what displays a set S itself. If we get $f_2(x) = Lf_1(x)$ from the condition $f_1(x) \in S$ - the operator L is stochastic. We will denote the set of stochastic operators by L_s .

Stochastic operator properties

1. Let $y = g(x)$ - differentiable monotonically increasing function on the axis Ox such that $\lim g(x) = -\infty, \lim g(x) = +\infty$.
The equation $y = g(x)$ has $x = h(y)$ as a solution that is differentiated and monotonically increasing on the axis Oy. Then the following functional relationship between $f_1(x)$ and $f_2(y)$ is true:

$$f_2(g(x))\frac{dg(x)}{dx} = f_1(x)$$

$$f_1(h(y))\frac{dh(y)}{dy} = f_2(y).$$

And corresponding stochastic operator L has the form

$$Lf(x) = f(h(x))\frac{dh(x)}{dx}$$

.

2. Let $x(t)$- random solution of the Cauchy problem for evolutionary systems of linear differential equations (1), with random initial conditions $x(t) = x_0$ has a density distribution $f(t, x)$. Then trues the equality

$$f(t, x) = L(t - \tau)f(\tau, x),$$

where the stochastic operator $L(t)$ is determined by the equality

$$L(t)f(x) = f(exp^{-AT}x)det(exp^{-At})$$

4.2. The analytical determination of Markov processes

In this part we briefly recall some properties of random Markov processes [82].

The random process is given as a series $\{x(t), t \in \mathbb{T}\}$ of random variables dependent on parameter t defined on a set $\mathbb{T}$, $\mathbb{T} \subset \mathbb{R}$. Parameter t is usually interpreted as time and it can be changing discretely or continuously. Due to this fact there exist discrete and continuous random processes.

Random process $\{\xi(t), t \in \mathbb{T}\}$, $\mathbb{T} \subset \mathbb{R}$, is called Markov if

$$P\{\xi(t) \le x|\xi(t_1), \xi(t_2), \dots, \xi(t_n)\} = P\{\xi(t) \le x|\xi(t_n)\}, \tag{28}$$

$$t_1 < t_2 < \ldots < t_n < t;\ x \in \mathbb{R}$$

holds where symbol $P\{A|B\}$ denotes conditional probability of event A given event B. Equality (28) is called Markov property of random process and is characteristic for all Markov processes. It can be interpreted as follows: the state to which the system is about to the move is determined only by the current state and not by states the system was in previously.

According to continuity of time and states we distinguish

- processes with discrete time and discrete states;
- processes with discrete time and continuous states;
- processes with continuous time and discrete states;
- processes with continuous time and continuous states.

Markov processes with discrete states are called Markov chains.

For discrete time Markov chain Definition looks as follows:
Series of discrete random variables

$$\{\xi_n,\ n = 0, 1, 2, \ldots\}, \quad \xi_n = \xi(n),$$

is called discrete time Markov's chain if for every n, $n = 0, 1, 2, \ldots$, and for every sequence x_0, x_1, …, x_n, x_{n+1} from a discrete set $\mathcal{X} \subset \mathbb{R}$ holds true

$$P\{\xi_{n+1} = x_{n+1}|\xi_0 = x_0, \xi_1 = x_1, \ldots, \xi_n = x_n\} = P\{\xi_{n+1} = x_{n+1}|\xi_n = x_n\}.$$

The set of all states we will denote $S = \{\theta_0, \theta_1, \theta_2, \ldots\}$ and we will assume that it is finite or countable.

Transition from one state to another is given by conditional probability called transition probability

$$\pi_{ij}(n+1) = P\left\{\xi_{n+1} = \theta_j|\xi_n = \theta_i\right\}. \tag{29}$$

If the system is in state θ_i at time moment n, then with probability $\pi_{ij}(n+1)$ it moves to state θ_j at moment $n+1$. If transition probabilities (29) depend only on states θ_i and θ_j and not on time parameter n, such chains are called homogenous and instead of $\pi_{ij}(n+1)$ we write π_{ij}. Otherwise, the chain is called non-homogeneous. Transition probabilities satisfy the following relationships

$$\begin{aligned} &0 \le \pi_{ij}(n) \le 1, \\ &\sum_{j=0}^{\infty} \pi_{ij}(n) = 1, \end{aligned} \qquad i, j = 0, 1, 2, \ldots.$$

When we write transition probabilities in the form of matrix, we get a matrix $\Pi(n)$ called transition probability matrix of Markov's chain or stochastic matrix

$$\Pi(n) = \begin{pmatrix} \pi_{00}(n) & \pi_{01}(n) & \pi_{02}(n) & \cdots \\ \pi_{10}(n) & \pi_{11}(n) & \pi_{12}(n) & \cdots \\ \pi_{20}(n) & \pi_{21}(n) & \pi_{22}(n) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

The rows represent the starting states and the columns represent the ending states during each unit of time. Properties of transition probabilities imply that the transition matrix $\Pi(n)$ must be nonnegative and the sum of every row is equal to 1.

The transition probability matrix determines the Markov chain except for the initial probability distribution

$$p_s(0) = P\left\{\xi_0 = \theta_s\right\}, \quad s = 0, 1, 2, \dots$$

that satisfy relationships

$$\begin{aligned} &0 \leq p_s(0) \leq 1, \\ &\sum_{s=0}^{\infty} p_s(0) = 1, \end{aligned} \qquad s = 0, 1, 2, \dots.$$

Probability distribution at any time moment is given by formula

$$\mathbf{p}(n+1) = \mathbf{p}(n)\Pi(n+1), \quad n = 0, 1, 2, \dots \tag{30}$$

where $\mathbf{p}(n) = (p_0(n), p_1(n), p_2(n), \dots)$, $n = 0, 1, 2, \dots$, denotes vector of probability distribution at n-th time moment and $\Pi(n+1)$ denotes transition probability matrix. If the chain is homogenous, the formula (30) can be rewritten in the form

$$\mathbf{p}(n+1) = \mathbf{p}(n)\Pi, \quad n = 0, 1, 2, \dots$$

or

$$\mathbf{p}(n+1) = \mathbf{p}(0)\Pi^n, \quad n = 0, 1, 2, \dots,$$

where $\Pi^n = \Pi.\Pi.\dots\Pi$.

Definition of continuous time Markov chain we write in the form:
Series of discrete random variables

$$\{\xi_t,\ t \geq 0\}, \quad \xi_t = \xi(t),$$

is called continuous time Markov chain if for all $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n \leq t \leq t + \Delta t$, and for all $x_1, \ldots, x_n, y, z$ from a discrete set $\mathcal{X} \subset \mathbb{R}$ holds true

$$P\{\xi_{t+\Delta t} = z | \xi_{t_1} = x_1, \xi_{t_2} = x_2, \ldots, \xi_{t_n} = x_n, \xi_t = y\} = P\{\xi_{t+\Delta t} = z | \xi_t = y\}.$$

For each $0 \leq t \leq t + \Delta t$ we can define transition probability matrix

$$\Pi(t, t + \Delta t) = (p_{ij}(t, t + \Delta t)),$$

where

$$p_{ij}(t, t + \Delta t) = P\{\xi_{t+\Delta t} = \theta_j | \xi_t = \theta_i\}, \quad i, j = 1, 2, \ldots \tag{31}$$

denotes transition probability from state θ_i at time t to state θ_j at time $t+\Delta t$.

Similarly as for discrete time Markov's chain probability vectors $\mathbf{p}(t)$ and $\mathbf{p}(t + \Delta t)$ are related by

$$\mathbf{p}(t + \Delta t) = \mathbf{p}(t)\Pi(t, t + \Delta t).$$

In special case that the transition matrix $\Pi(t, t+\Delta t)$ depends only on the time difference $t + \Delta t - t = \Delta t$ that is $\Pi(t, t + \Delta t) = \Pi(0, \Delta t)$ for $0 \leq t \leq t + \Delta t$, the continuous time Markov's chain is homogeneous.

Now we assume that for transition probabilities (31) there exist limits

$$\begin{aligned} \lim_{\Delta t \to 0} p_{ij}(t + \Delta t) &= 0, \quad i \neq j, \\ \lim_{\Delta t \to 0} p_{ij}(t + \Delta t) &= 1, \quad i = j. \end{aligned}$$

This implies that transition probability $p_{ij}(t, t + \Delta)$ at any time moment t is equal 0. Therefore it is more convenient to use transition intensity instead of transition probability. Let there exist limits

$$\begin{aligned} \lim_{\Delta t \to 0} \frac{p_{ij}(t, t + \Delta t)}{\Delta t} &= \gamma_{ij}(t), \\ \lim_{\Delta t \to 0} \frac{1 - p_{ii}(t, t + \Delta t)}{\Delta t} &= \gamma_{ii}(t), \end{aligned}$$

where $\gamma_{ij} \geq 0$ for all i, j. Functions $\gamma_{ij}(t)$ are called transition probabilities and they define the probability intensity of transition from state θ_i to state θ_j at time t or in case that $i = j$ probability intensity of remain in state θ_i at time t. If the continuous time Markov chain is homogeneous, the transition intensities are not functions depending on t but constants.

We can write the transition intensities $\gamma_{ij}(t)$ in the form of matrix

$$\Gamma(t) = \begin{pmatrix} -\gamma_{11}(t) & \gamma_{12}(t) & \dots & \gamma_{1r}(t) & \dots \\ \gamma_{21}(t) & -\gamma_{22}(t) & \dots & \gamma_{2r}(t) & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \gamma_{r1}(t) & \gamma_{2r}(t) & \dots & -\gamma_{rr}(t) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

called the transition intensity matrix. Sum of each row has to be equal 0. The homogeneous continuous time Markov's chain is completely characterized by the transition intensity matrix $\Gamma(t)$ and the initial probability vector $\mathbf{p}(0)$.

The probability distribution given as probability vector $\mathbf{p}(t)$ in any time t can be found as a solution of differential system

$$\frac{d\,\mathbf{p}(t)}{d\,t} = \mathbf{p}(t)\Gamma(t), \quad t \geq 0$$

or in case that the continuous time Markov's chain is homogeneous

$$\frac{d\,\mathbf{p}(t)}{d\,t} = \mathbf{p}(t)\Gamma, \quad t \geq 0.$$

4.3. Moments of Random Variable

This section is processed according to [71].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where Ω is sample space, $\mathcal{F}$ is set of all possible events (the σ-algebra) and $\mathbb{P}$ is some probability measure. Let us consider a system of m continuous random variables $(x_1, x_2, \dots, x_m)$ characterized by density function $f\,(x_1, x_2, \dots, x_m)$ satisfying conditions

$$f\,(x_1, x_2, \dots, x_m) \geq 0,$$
$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f\,(x_1, x_2, \dots, x_m)\, dx_1 \dots dx_m = 1.$$

The given conditions could be rewritten in vector form

$$f\,(x) \geq 0,$$
$$\int_{\mathbb{E}_m} f\,(x)\, dx = 1,$$

where $\mathbb{E}_m$ denotes m-dimensional Euclidean space, $x = (x_1, x_2, \dots, x_m)$ denotes a column vector of random variables $x_1, x_2, \dots, x_m$, $dx \equiv dx_1 \dots dx_m$.

Now we consider system of random variables $(x_1, x_2 \dots, x_m, \xi)$, where $x_1, x_2 \dots, x_m$ are continuous random variables and ξ is discrete random variable which gains values $\theta_1, \dots, \theta_q$. The density function of random variable $(x_1, x_2 \dots, x_m, \xi)$ is given as

$$f(x, \xi) = f(x_1, x_2 \dots, x_m, \xi) = \sum_{k=1}^{q} f_k(x)\delta(\xi - \theta_k),$$

where $\delta(\xi)$ is Dirac delta function. Functions $f_k(x) \equiv f_k(x_1, x_2 \dots, x_m)$, $k = 1, 2, \dots, n$, that satisfy conditions

$$f_k(x) \geq 0, \quad \int_{\mathbb{E}_m} \sum_{k=1}^{q} f_k(x) dx = 1,$$

$x \in \mathbb{E}_m$, $k = 1, 2, \dots, q$, are called particular density functions because they represent parts of the density function

$$f(x) = \sum_{k=1}^{q} f_k(x).$$

Let $x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m$ be the system of random variables depending on the random Markov process ξ with q possible states θ_k, $k = 1, 2, \dots, q$. The matrices

$$E^{(1)}\{x\} = \sum_{k=1}^{q} E_k^{(1)}\{x\}, \quad E^{(2)}\{x\} = \sum_{k=1}^{q} E_k^{(2)}\{x\},$$

where

$$E_k^{(1)}\{x\} = \int_{\mathbb{E}_m} x\, f_k(x)\, dx, \quad E_k^{(2)}\{x\} = \int_{\mathbb{E}_m} xx^* f_k(x)\, dx,$$

$k = 1, 2, \dots, q$, are called moments of the first or the second order of the vector random variable x respectively. The values $E_k^{(1)}\{x\}$ and $E_k^{(2)}\{x\}$, $k = 1, 2, \dots, q$, are called particular moments of the first or the second order respectively. Symbol $*$ denotes the Hermitian transpose of matrix. First moment $E^{(1)}\{x\}$ of random variable x is also known as a mathematical expectation of x and it is usually denoted as $E(x)$ or $\langle x \rangle$. The moments of the random variable x in the scalar case, $x \in \mathbb{R}$, are defined for any $s = 1, 2, \dots$, and are called moments of the s-th order. The particular moments are defined by the formula

$$E_k^{(s)}\{x(t)\} = \int_{-\infty}^{\infty} x^s f_k(t, x)\, dx, \quad s = 1, 2, \dots, \quad k = 1, 2, \dots, q. \tag{32}$$

4.4. Moment Equations for non-homogeneous differential equations.

We have derived the system of moment equations for a linear homogenous equation with random coefficient under assumptions that the random variable can only be in two states. It was a simple enough case that allowed us to understand the process of deriving the system of moment equations. Now we establish a system of moment equations in the same way for the linear non-homogeneous differential equation

$$\frac{dx(t)}{dt} = a(t, \xi(t))x(t) + b(t, \xi(t)), \tag{33}$$

where $\xi(t)$ is the continuous time Markov's chain which has q possible states $\theta_1, \theta_2, \ldots, \theta_q$ with probabilities $p_k(t) = P\{\xi(t) = \theta_k\}$, $k = 1, 2, \ldots, q$. We suppose that the probabilities satisfy the system of linear differential equations

$$\frac{dp_k(t)}{dt} = \sum_{s=1}^{q} \gamma_{ks}(t)p_s(t), \tag{34}$$

where the transition intensity matrix $(\gamma_{ks}(t))_{k,s=1}^{q}$ satisfies the following relationships:

$$\sum_{k=1}^{q} \gamma_{sk}(t) \equiv 0, \quad \gamma_{ks}(t) \begin{cases} \geq 0, & k \neq s, \\ \leq 0, & k = s. \end{cases}$$

Since the coefficients of studied system (33) depend on t, we can denote

$$a_k(t) = a(t, \theta_k), \quad b_k(t) = b(t, \theta_k), \quad k = 1, 2, \ldots, q.$$

Theorem 4. Moment equations of any order $s = 1, 2, \ldots$ for differential equation(33) are of the form

$$\frac{dE_s^{(k)}\{x(t)\}}{dt} = s\, a_k(t)E_s^{(k)}\{x(t)\} + s\, b_k(t)E_{s-1}^{(k)}\{x(t)\} + \tag{35}$$

$$+ \sum_{r=1}^{q} \gamma_{rk}(t)E_s^{(r)}\{x(t)\},$$

$$k = 1, 2, \ldots, q.$$

4.5. Moment Equations for the linear differential dystems.

Now we consider the Linear Differential System

$$\frac{dx(t)}{dt} = A(t,\xi(t))x(t) + B(t,\xi(t)). \tag{36}$$

We also suppose that the matrix A and the vector B depend on a random continuous time Markov's chain $\xi(t)$ with q possible states, the probabilities of which satisfy the system of linear differential equations (24). Moreover, we use the denotations

$$A_k(t) = A(t,\theta_k), \quad B_k(t) = B(t,\theta_k), \quad k = 1,2,\ldots,q.$$

Theorem 5. Moment equations of the first and second order respectively for system (36) are of the form

(37)

$$\frac{dE_k^{(1)}\{x(t)\}}{dt} = A_k(t)E_k^{(1)}\{x(t)\} + B_k(t)E_k^{(0)}\{x(t)\} + \sum_{j=1}^{q}\gamma_{jk}(t)E_j^{(1)}\{x(t)\},$$

$$\frac{dE_k^{(2)}\{x(t)\}}{dt} = A_k(t)E_k^{(2)}(t) + E_k^{(2)}\{x(t)\}A_k^*(t) + B_k(t)\left(E_k^{(1)}\{x(t)\}\right)^*$$

$$+E_k^{(1)}\{x(t)\}B_k^*(t) + \sum_{j=1}^{q}\gamma_{jk}(t)E_j^{(2)}\{x(t)\}, k = 1,2,\ldots,q. \tag{38}$$

Proof. The philosophy of the proof is the same as in the proof of Theorem 4, only the calculations are more complicated because now we work with the matrix case. In a similar way, by dividing the time line into intervals of length h for the particular density functions $f_k(t,x)$, $k = 1,2\ldots,q$, we get the system of equations

$$f_k(t_{n+1},x) = f_k(t_n,Y_k)\upsilon_k + h\sum_{j=1}^{q}\gamma_{jk}(t_n)f_j(t_n,Y_j)\upsilon_j, \tag{39}$$

$$k = 1,2,\ldots,q,$$

where

$$\begin{aligned} Y_j &= (I + hA_j(t_n))^{-1}(x - hB_j(t_n)), \\ \upsilon_j &= \det(I + hA_j(t_n))^{-1}, \end{aligned} \qquad j = 1,2,\ldots,q,$$

and I is the unit matrix.

Assume that the particular density functions can be expressed in powers of parameter h by the Taylor's formula. If we put $t_n = t$, then the decompositions of the functions on the left-hand side and on the right-hand side in (39) are equal to

$$\begin{aligned} Y_j &= x - h[A_j(t)x + B_j(t)] + \mathcal{O}(h^2), \\ \upsilon_j &= \det(I - hA_j(t) + \mathcal{O}(h^2)) = 1 - h\,Tr(A_j(t)) + \mathcal{O}\left(h^2\right), \end{aligned}$$

$$\begin{aligned} f_k(t_{n+1}, x) &= f_k(t+h, x) = f_k(t,x) + \frac{\partial f_k(t,x)}{\partial t} + \mathcal{O}\left(h^2\right), \\ f_k(t, Y_k) &= f_k\left[t, x - h(A_k(t)x + B_k(t)) + O\left(h^2\right)\right] = \\ &= f_k(t,x) - grad\, f_k(t,x)\,[A_k(t)x + B_k(t)]\,h + O\left(h^2\right), \end{aligned}$$

$k = 1, 2, \dots, q$,

where $Tr(A)$ is the trace of the matrix A, $\dfrac{d}{dt}\det\left(A_j(t)\right) = Tr\left(A_j(t)\right)$ and

$$grad\, f(t,x) = \left(\frac{\partial f(t,x)}{\partial x_1}, \frac{\partial f(t,x)}{\partial x_2}, \cdots, \frac{\partial f(t,x)}{\partial x_m}\right).$$

Using obtained expressions, comparing the left-hand side to the right-hand side of equation (39) and assuming $h \to 0$, we get the system of differential equations for the particular density functions

$$\begin{aligned} \frac{\partial f_k(t,x)}{\partial t} &= -f_k(t,x)Tr(A_k(t)) - grad\, f_k(t,x)\,[A_k(t)x + B_k(t)] \\ &\quad - \sum_{j=1}^{q} \gamma_{jk}(t) f_j(t,x), \quad k = 1, 2, \dots, q. \end{aligned} \tag{40}$$

Finally, multiplying equation (40) by x and integrating it by parts of the Euclidean space $\mathbb{E}_m$, in accordance by Definition of Moments, we obtain a system of linear equations for the particular moments of the first order in the form (37). The particular moments of second order satisfy the matrix system of differential equations (38) which we get in the same way. The difference is that (40) is multiplied by the matrix, xx^*, next it is integrated over the Euclidean space $\mathbb{E}_m$. □

The moment equations (37), (38) are deterministic and can be solved by using usual methods.

5. Concluding productions problems. We set the task of researching stabilization classes dynamical systems with random parameters. The results obtained for these classes systems:
1. The class of linear differential equations with random coefficients and random jumps of solutions.
2. The class of linear difference equations with random coefficients and random jumps of solutions.

6. Hope. We will investigate in the future:
1. The class of nonlinear difference equations with random coefficients and random transformations of solutions.
2. The class of nonlinear differential equations with random coefficients and random transformations of solutions.
3. The class of stochastic differential equations of a special type and so on.

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