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ON POSITIVE SOLUTIONS OF A TWO-POINT BOUNDARY VALUE PROBLEM FOR A CLASS OF HIGHER-ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this paper a family of two-point boundary value problems is considered for a class of nonlinear ordinary differential equations $y^{(n)} + x^m|y|^k = 0$. We study properties of solutions for the class of equations and prove a theorem on the existence and uniqueness of a positive on $(0, 1)$ solution to such boundary value problems. This solution is presented via a solution of a related initial-value problem.

Key Words. existence and uniqueness of a positive solution, a two-point nonlinear boundary value problem.

AMS(MOS) subject classification. 34B18

Introduction. A lot of mathematical works devoted to boundary value problems has been published, some of them are [1]–[12]. Some actual questions in the theory of boundary value problems of ordinary differential equations are studied in [10] with methods proposed for numerical solving applied problems. Na T. described the main numerical methods for solving two-point boundary value problems in his monograph [6].

Bearing in mind that analytic finding solutions to nonlinear boundary value problems is usually difficult, there are various approaches to study solvability of problems. Different types of solutions of two-point nonlinear boundary value problems are investigated and the number of solutions is estimated in [3]. F. J. Sadyrbaev and I. R. Yermachenko have obtained a result on the existence and multiplicity of solutions of a two-point boundary value problem for an Emden-Fowler equation by the quasilinearization in [11,

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12]. The existence and uniqueness of positive solutions of a singular fourth-order boundary value problems are considered using the mixed monotone method in [4]. An important role in the study of various classes of nonlinear problems was acquired by the methods of differential inequalities and fixed points (for example, in [9]). A constructive approach based on the method of differential inequalities was proposed by M. Nagumo in [7] and is a common method for obtaining sufficient conditions for the existence of solutions of a boundary value problem. The method of upper and lower solutions is used in [8] to study the boundary value problem with impulse action.

Particular attention is paid to positive solutions of boundary value problems in the papers [1], [4]–[5]. In [5] methods of proving the existence of positive solutions are stated and conditions of the uniqueness of the positive solution are indicated. The research presented here extends some conclusions about the existence of a unique positive solution of the two-point boundary value problem for a fourth-order equation (obtained by E. I. Abduragimov in [1]) to the case of an equation of arbitrary order. The methods described by I. V. Astashova in the book [2, chapters 5,6] have been used to prove the results.

1. Statement of the problem. Consider the family of two-point boundary value problems

$$y^{(n)}(x) + x^m |y(x)|^k = 0, \tag{1}$$

$$y(0) = y'(0) = \ldots = \widehat{y^{(j)}(0)} = \ldots = y^{(n-1)}(0) = 0, \tag{2}$$

$$y^{(i)}(1) = 0, \tag{3}$$

where each problem from the family is set by fixing parameters $n \geqslant 2$, $m \geqslant 0$, $k > 1$, $0 \leqslant i \leqslant j \leqslant n-1$. At the point $x = 0$, the values of all derivatives up to the $(n-1)$-th one are known, except for j-th, while at the point $x = 1$ one boundary condition is set for the derivative of order i. Thus, we have n boundary conditions.

We consider the existence of solution that is positive on $(0, 1)$. The more general case with the interval $(0, a)$ reduces to that one.

2. Related initial-value problem. To study the above boundary value problem (1)-(3) consider a related initial-value problem, namely

$$z^{(n)}(x) + x^m |z(x)|^k = 0, \tag{4}$$

$$z(0) = z'(0) = \ldots = \widehat{z^{(j)}(0)} = \ldots = z^{(n-1)}(0) = 0, \tag{5}$$

$$z^{(j)}(0) = A \tag{6}$$

with the same n, m, k, j and an arbitrary number $A > 0$.

For this problem, the equation is the same as for the boundary problem (1)-(3) introduced in Section 1. The initial conditions are set at zero so that all derivatives but one are zero and a single derivative is positive.

3. Auxiliary statements.

LEMMA 1. *Let $z(x)$ be a solution to the initial-value problem (4)–(6) maximally extended to the right. Then there exist uniquely defined points $x_0 > x_1 > \ldots > x_j > 0$ such that for each $l = 0, \ldots, j$ the derivative $z^{(l)}$ vanishes at x_l, is positive on $(0, x_l)$, and is negative on (x_l, x^*), where $x^* \leqslant +\infty$ is the right-hand end-point of the domain of z.*

Proof. Integrate (4) with respect to s between 0 and x:

$$\int_0^x (z^{(n)}(s) + s^m|z(s)|^k)\,ds = z^{(n-1)}(x) - z^{(n-1)}(0) + \int_0^x s^m|z(s)|^k\,ds = 0.$$

We obtain an expression for $z^{(n-1)}(x)$:

$$z^{(n-1)}(x) = z^{(n-1)}(0) - \int_0^x s^m|z(s)|^k\,ds. \tag{7}$$

Taking into account initial conditions (5)–(6), we admit two possible situations:

$$z^{(n-1)}(x) = \begin{cases} A - \int_0^x s^m|z(s)|^k\,ds & \text{if } j = n-1, \\ -\int_0^x s^m|z(s)|^k\,ds & \text{if } j \neq n-1. \end{cases}$$

Next, integrate both sides of (7):

$$\begin{aligned} z^{(n-2)}(x) &= z^{(n-2)}(0) + xz^{(n-1)}(0) - \int_0^x \Big(\int_0^t s^m|z(s)|^k\,ds\Big)\,dt \\ &= z^{(n-2)}(0) + xz^{(n-1)}(0) - \int_0^x \Big(\int_s^x s^m|z(s)|^k\,dt\Big)\,ds \\ &= z^{(n-2)}(0) + xz^{(n-1)}(0) - \int_0^x (x-s)s^m|z(s)|^k\,ds. \end{aligned}$$

In a similar way, there are different expressions for $z^{(n-2)}(x)$ in relation to j:

$$z^{(n-2)}(x) = \begin{cases} Ax - \int_0^x (x-s)s^m|z(s)|^k\,ds, & \text{if } j = n-1, \\ A - \int_0^x (x-s)s^m|z(s)|^k\,ds, & \text{if } j = n-2, \\ -\int_0^x (x-s)s^m|z(s)|^k\,ds, & \text{if } j \notin \{n-1, n-2\}. \end{cases}$$

If we continue to integrate successfully, we will get expressions for all derivatives of $z(x)$:

$$z^{(l)}(x) = \begin{cases} -\int_0^x \frac{(x-s)^{(n-l-1)}}{(n-l-1)!} s^m |z(s)|^k\, ds, & \text{if } j+1 \leqslant l \leqslant n-1, \\ A\frac{x^{(j-l)}}{(j-l)!} - \int_0^x \frac{(x-s)^{(n-l-1)}}{(n-l-1)!} s^m |z(s)|^k\, ds, & \text{if } 0 \leqslant l \leqslant j. \end{cases}$$

Note that $z^{(l)}(x) \leqslant 0$ when $x \in (0, x^*)$, $j+1 \leqslant l \leqslant n-1$. It follows herefrom that all $z^{(j)}(x), \ldots, z^{(n-1)}(x)$ decrease. In particular, if $j \neq n-1$, then $z^{(j)}(x)$ monotonically decreases and is positive at zero according to initial condition (6).

When $j = n-1$, it is necessary to check that $z^{(n)}(x) \leqslant 0$. Indeed, from the equation (4) we have

$$z^{(n)}(x) = -x^m |z(x)|^k \leqslant 0, x \in (0, x^*).$$

Consider the case when $x^* < \infty$ and the solution $z(x)$ cannot be extended over x^*. In order to prove the existence of a point where vanishes the derivative $z^{(j)}$, it suffices to show that $z^{(j)}(x)$ cannot have a non-negative limit as $x \to x^*$.

Assume the converse, i. e. $z^{(j)}(x) \to c \geqslant 0$ as $x \to x^*$. Then in the case $j > 0$, the derivative $z^{(j-1)}(x)$ tends to some non-negative constant:

$$z^{(j-1)}(x) = z^{(j-1)}(0) + \int_0^x z^{(j)}(s)\, ds = \int_0^x z^{(j)}(s)\, ds,$$

$$\lim_{x \to x^*} z^{(j-1)}(x) = \lim_{x \to x^*} \int_0^x z^{(j)}(s)\, ds.$$

This implies the similar statement for the derivative $z^{(l)}(x)$ of any order l with $0 \leqslant l \leqslant j$. From equation (4) we have $z^{(n)}(x) = -x^m |z(x)|^k$. Since $z(x)$ tends to a non-negative constant, $z^{(n)}(x)$ tends to a non-positive limit. Taking into account that

$$z^{(n-1)}(x) = z^{(n-1)}(0) + \int_0^x z^{(n)}(s)\, ds,$$

$$\lim_{x \to x^*} z^{(n-1)}(x) = z^{(n-1)}(0) + \lim_{x \to x^*} \int_0^x z^{(n)}(s)\, ds,$$

$z^{(n-1)}(x)$ also tends to a constant. And this statement is true for the derivatives $z^{(n-2)}(x), \ldots, z^{(j+1)}(x)$.

In this way, if $x^* < \infty$ and at least one of the limits, for example, this of $z^{(j)}(x)$ is finite, then the limits of the functions $z^{(j-1)}(x), \ldots, z(x)$ are also finite as well as the limits of $z^{(n)}(x), \dots, z^{(j+1)}(x)$, which contradicts the non-extencibility of the solution over x^*. This means that $z^{(j)}(x)$ cannot tend to a non-negative constant and therefore vanishes at some point denoted by x_j.

Pass on to the case $x^* = +\infty$. It is neessary to show that there exists a point $x_j \in (0, +\infty)$ with $z^{(j)}(x_j)$. Assume the converse, i. e. $z^{(j)}(x) > 0$ whenever $x \in (0, +\infty)$.

Consider the case $j = n-1$. If $z^{(j)}(x) > 0$ on the $(0, +\infty)$, then $z^{(j-1)}(x)$ increases and, at the same time, $z^{(j-1)}(0) = 0$. It means that $z^{(j-1)}(x)$ tends to a positive limit (finite or infinite). Therefore all $z^{(j-2)}(x)$, ..., $z(x)$ tend to positive limits. Hence, $z^{(n)}(x) \to -\infty$ as $x \to \infty$, whence $z^{(n-1)}(x) \to -\infty$, which means that $z^{(n-1)}(x)$ cannot remain positive everywhere and vanishes at some point.

If $j < n-1$, then choose a point $x_0 > 0$ and integrate twice from x_0 to x the obtained already inequality $z^{(j+2)}(x) \le 0$:

$$z^{(j+1)}(x) \le z^{(j+1)}(x_0),$$

$$z^{(j)}(x) \le z^{(j)}(x_0) + z^{(j+1)}(x_0)(x - x_0).$$

Since $z^{(j+1)}(x_0) < 0$ and $(x - x_0) \to +\infty$ as $x \to +\infty$, we obtain by going to the limit that $z^{(j)}(x) \to -\infty$, which contradicts to the assumption that $z^{(j)}(x) > 0$ on $(0, +\infty)$.

Hence $z^{(j)}(x)$ vanishes at a unique point of $(0, +\infty)$. We denote it by x_j. The derivative $z^{(j)}$ must be positive on $(0, x_j)$ and negative on $(x_j, +\infty)$.

Now consider the derivative of the lower order. According to the initial conditions, $z^{(j-1)}(0) = 0$. Since $z^{(j)}(x) > 0$ when $x \in (0, x_j)$ and $z^{(j)}(x) < 0$ when $x \in (x_j, +\infty)$, the function $z^{(j-1)}(x)$ increases before the point x_j and decreases after it. Similar reasoning shows that, considering separately the cases $x^* = \infty$ and $x^* < \infty$, there exists a unique point x_{j-1} where the function $z^{(j-1)}$ changes its sign from positive to negative.

The same reasoning is appropriate for the existence proof of points $x_{j-2}, \dots, x_0$. ∎

Fig. 1 below generally illustrates behavior of all derivatives $z'(x), \dots, z^{(n-1)}(x)$ and of the function $z(x)$ itself.

It should be noted that the solution $z(x)$ satisfies conditions (1)–(2). It remains to transform this solution to meet condition (3).

Lemma 2. *Let $z(x)$ be a positive on $(0, \alpha)$ solution to equation* (1). *If $C > 0$ and*

$$B = C^{\frac{n+m}{k-1}}, \tag{8}$$

then $y(x) = Bz(Cx)$ is a positive on $(0, \frac{\alpha}{C})$ solution to equation (1).

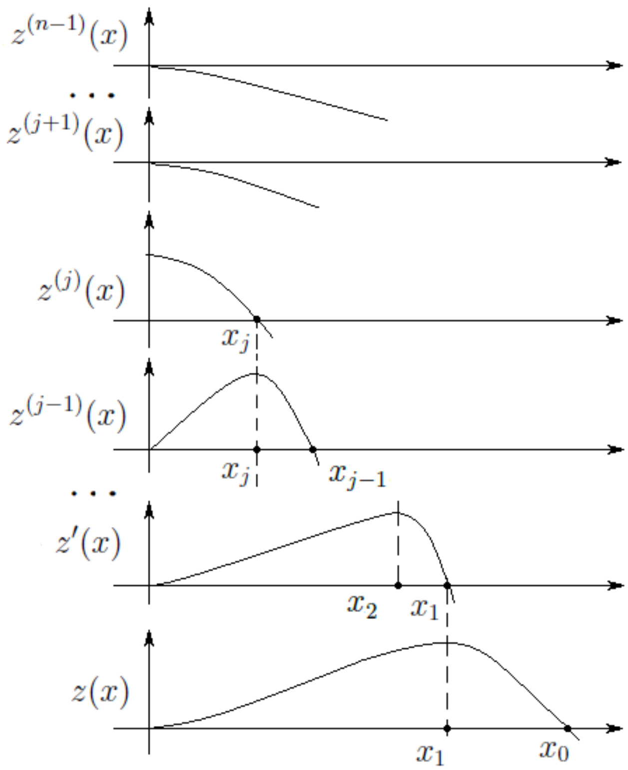

FIG. 1. *General behavior of derivatives $z'(x), \dots, z^{(n-1)}(x)$ and $z(x)$ itself.*

Proof. First, we find the n-th derivative of $y(x)$:

$$
\begin{aligned}
&y'(x) = BCz'(Cx),\\
&y''(x) = BC^2z''(Cx),\\
&\dots\\
&y^{(n)}(x) = BC^nz^{(n)}(Cx).
\end{aligned}
$$

Next we put the expressions for the function $y(x)$ and its n-th derivative into equation (1)

$$
\begin{aligned}
&BC^nz^{(n)}(Cx) + x^m|Bz(Cx)|^k = 0,\\
&z^{(n)}(Cx) + x^mB^{k-1}C^{-n}|z(Cx)|^k = 0,\\
&z^{(n)}(Cx) + (Cx)^mB^{k-1}C^{-n-m}|z(Cx)|^k = 0.
\end{aligned}
$$

By using (8), we obtain the valid statement since $z(x)$ is a solution of equation (1):

$$z^{(n)}(Cx) + (Cx)^m C^{\frac{(n+m)(k-1)}{k-1} - (n+m)} |z(Cx)|^k = 0,$$
$$z^{(n)}(Cx) + (Cx)^m |z(Cx)|^k = 0.$$

From the condition $z(x) > 0$ on $(0, \alpha)$ it follows that $z(Cx) > 0$ as well as $y(x) > 0$ on $x \in (0, \frac{\alpha}{C})$. ∎

4. Main result.

THEOREM 1. *There exists a positive on* $(0,1)$ *solution* y *to the two-point boundary value problem* (1)–(3). *The solution is uniquely defined by the formula*

$$y(x) = Bz(Cx),$$

where $z(x)$ *is a solution of the related initial-value problem* (4)–(6)*, the constant* C *is equal to* x_i *provided by Lemma 1, and* B *is defined by* (8).

Proof. It follows from Lemma 1 that there exist some unique points x_l, $l = 0, \dots, j$, such that the solution $z(x)$ of the initial-value problem (4)–(6) can be defined uniquely on $[0, x_0]$ and is positive on the interval $(0, x_0)$. Besides, $z^{(l)}(x_l) = 0$ for each $l = 0, \dots, j$.

By using the property of solutions to equation (1) provided by Lemma 2, we can show that $y(x) = Bz(Cx)$ is a solution of the differential equation considered and satisfies conditions (2).

Now we find its derivative of order i:

$$y^{(i)}(x) = BC^i z^{(i)}(Cx).$$

Thus we have $y^{(i)}(1) = BC^i z^{(i)}(C)$ and therefore condition (3) is satisfied whenever $C = x_i$. ∎

Conclusions. In the present paper it was proved that each two-point boundary value problems (1)–(3) has a unique positive solution on $(0,1)$.

In the further research, it is planned to determine the class of functions $f(x)$ such that the results will be correct for the new family of boundary value problems

$$y^{(n)}(x) + f(x)|y(x)|^k = 0,$$
$$y(0) = y'(0) = \dots = \widehat{y^{(j)}(0)} = \dots = y^{(n-1)}(0) = 0,$$
$$y^{(i)}(1) = 0,$$

where $n \geqslant 2$, $k > 1$, $0 \leqslant j \leqslant n-1$, $0 \leqslant i \leqslant j$.

For example, it can be shown that the results are generalized to the case when $f(x)$ is a polynomial of degree m and $f(x) \geqslant 0$ on $(0, \gamma)$.

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