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OSCILLATION CRITERIA FOR EVEN-ORDER DIFFERENTIAL EQUATIONS WITH UNBOUNDED NEUTRAL COEFFICIENTS AND DISTRIBUTED DEVIATING ARGUMENTS

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Abstract. This paper is concerned with the oscillatory behavior of solutions of a class of even-order differential equations with unbounded neutral coefficients and distributed deviating arguments. New sufficient conditions for the oscillation of all solutions are established that are not covered by existing results in the literature. The results are illustrated with some examples.

Key Words. Oscillation, even-order, neutral differential equation, distributed deviating arguments, comparison theorem

AMS(MOS) subject classification. 34C10, 34C15, 34K11

1. Introduction . The main objective of this paper is to establish some new criteria for the oscillation of all solutions of the even-order neutral differential equation with distributed deviating arguments

$$z^{(n)}(t) + \int_a^b q(t,\xi)x^{\delta}\left(\varphi(t,\xi)\right)d\xi = 0, \tag{1}$$

where $z(t) = x(t) + p(t)x(\tau(t))$, $t \geq t_0 > 0$, $0 < a < b$, and $\delta > 0$ is the quotient of odd positive integers. In the remainder of the paper we assume that:

(i) $p : [t_0, \infty) \to \mathbb{R}$ is a real-valued continuous function with $p(t) \geq 1$, and $p(t) \not\equiv 1$ for large t;

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(ii) $q : [t_0, \infty) \times [a, b] \to (0, \infty)$ is a real-valued continuous function;
(iii) $\tau : [t_0, \infty) \to \mathbb{R}$ is a real-valued continuous function such that $\tau(t) \geq t$, τ is strictly increasing, and $\lim_{t\to\infty} \tau(t) = \infty$;
(iv) $\varphi : [t_0, \infty) \times [a, b] \to \mathbb{R}$ is nonincreasing in ξ, and

$$\lim_{t\to\infty} \varphi(t, \xi) = \infty, \quad \xi \in [a, b].$$

By a *solution* of equation (1), we mean a function $x : [t_x, \infty) \to \mathbb{R}$, $t_x \geq t_0$, such that $x(t) + p(t)x(\tau(t)) \in C^n([t_x, \infty), \mathbb{R})$ and that satisfies equation (1) on $[t_x, \infty)$. We consider only those solutions $x(t)$ of (1) that satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq t_x$; and moreover, we tacitly assume that (1) possesses such solutions. Such a solution $x(t)$ of (1) is said to be *oscillatory* if it has arbitrarily large zeros on $[t_x, \infty)$, i.e., for any $t_1 \in [t_x, \infty)$ there exists a $t_2 \geq t_1$ such that $x(t_2) = 0$; otherwise it is called *nonoscillatory*, i.e., if it is eventually positive or eventually negative. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

Recently, the problem of obtaining sufficient conditions to ensure that all solutions to various classes of even-order neutral differential equations are oscillatory has been studied by a number of authors. For typical results, we refer the reader to the papers [2, 4, 5, 6, 14, 20, 25, 27] and the references cited therein. However, determining oscillation criteria for even-order neutral differential equations with distributed deviating arguments has not received a great deal of attention in the literature; moreover, the results obtained are for the cases $0 \leq p(t) \leq p_0 < 1$ and/or $-1 < p(t) \leq 0$. This means that the results obtained in these papers cannot be applied to the case where $p(t) \to \infty$ as $t \to \infty$; for example, see the papers [1, 9, 11, 12, 15, 16, 17, 18, 21, 22, 23, 24, 26] and the references contained therein as examples of recent results on this topic. For these reasons, here we wish to establish some new oscillation criteria for the case $p(t) \to \infty$ as $t \to \infty$. We note that the results presented in this paper are new even if $\delta = 1$ and can easily be extended to more general even-order neutral differential equations with distributed deviating arguments (see Remark 2 below). The results here compliment those in [10, 13]. It is our hope that the present paper will stimulate additional interest in research on even-order neutral differential equations in general, and those with unbounded neutral coefficients and distributed deviating arguments in particular.

2. Main results . We begin with the following lemmas that will play an important role in establishing our main results. For notational purposes, we let

$$\varphi_1(t) := \varphi(t, b) \quad \text{and} \quad \psi(t) := \frac{1}{p(\tau^{-1}(t))} \left(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\right),$$

where τ^{-1} is the inverse of τ. We assume throughout that $\psi(t) > 0$ for all sufficiently large t, and we set

$$Q(t) := \int_a^b q(t,\xi)\psi^\delta(\varphi(t,\xi))d\xi.$$

LEMMA 1 (Kiguradze [7]). *Let $f \in C^n([t_0,\infty),(0,\infty))$. If the derivative $f^{(n)}(t)$ is eventually of one sign for all large t, then there exist $t_x \geq t_0$ and an integer l, $0 \leq l \leq n$, with $n+l$ even for $f^{(n)}(t) \geq 0$, or $n+l$ odd for $f^{(n)}(t) \leq 0$, such that*

$$l > 0 \quad \textit{implies that } f^{(k)}(t) > 0 \quad \textit{for } t \geq t_x, \quad k = 0,1,...,l-1,$$

and

$$l \leq n-1 \quad \textit{implies that } (-1)^{l+k} f^{(k)}(t) > 0 \quad \textit{for } t \geq t_x, \quad k = l, l+1, ..., n-1.$$

LEMMA 2 ([3, Lemma 2.2.3]). *Let $f \in C^n([t_0,\infty),(0,\infty))$, $f^{(n)}(t)f^{(n-1)}(t) \leq 0$ for $t \geq t_x$ for some $t_x \geq t_0$, and assume that $\lim_{t\to\infty} f(t) \neq 0$. Then for each $\lambda \in (0,1)$, there exists $t_\lambda \in [t_x,\infty)$ such that, for all $t \in [t_\lambda,\infty)$,*

$$f(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} \left| f^{(n-1)}(t) \right|. \tag{2}$$

THEOREM 1. *Let conditions (i)–(iv) hold and assume that $\varphi(t,\xi) \leq \tau(t)$ for $(t,\xi) \in [t_0,\infty) \times [a,b]$. If there exists $\lambda_0 \in (0,1)$ such that the first-order delay differential equation*

$$y'(t) + \frac{\lambda_0}{((n-1)!)^\delta} Q(t) \left(\tau^{-1}(\varphi_1(t))\right)^{\delta(n-1)} y^\delta\left(\tau^{-1}(\varphi_1(t))\right) = 0 \tag{3}$$

is oscillatory, then equation (1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of equation (1). Without loss of generality, we may assume that there exists $t_1 \in [t_{0,}\infty)$ such that $x(t) > 0$ for $t \in [t_1,\infty)$, $x(\varphi(t,\xi)) > 0$ for $(t,\xi) \in [t_1,\infty) \times [a,b]$, and $\psi(t) > 0$ for $t \geq t_1$. The proof if $x(t)$ is eventually negative is similar, so we omit the details of that case here as well as in the remaining proofs in this paper. Then, in view of (i)–(iv), there exists $t_2 \geq t_1$ such that

$$z(t) > 0 \quad \text{and} \quad z^{(n)}(t) < 0 \quad \text{for} \quad t \geq t_2.$$

Since n is even, from Lemma 1 there exists $t_3 \geq t_2$ such that

$$z'(t) > 0 \quad \text{and} \quad z^{(n-1)}(t) > 0 \quad \text{for} \quad t \geq t_3.$$

Since $z(t) > 0$ and $z'(t) > 0$, we have $\lim_{t\to\infty} z(t) \neq 0$. Choose $\lambda_0 \in (0,1)$ and $t_{\lambda_0} \geq t_3$ such that equation (3) is oscillatory and condition (2) holds for $z(t)$ and $t \geq t_{\lambda_0}$, i.e.,

$$z(t) \geq \frac{\lambda_0}{(n-1)!} t^{n-1} z^{(n-1)}(t) \quad \text{for} \quad t \geq t_{\lambda_0}. \tag{4}$$

Then for $t \geq t_{\lambda_0}$, it follows that

$$\begin{aligned} x(t) &= \frac{1}{p(\tau^{-1}(t))}\left[z(\tau^{-1}(t)) - x(\tau^{-1}(t))\right] \\ &= \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{[z(\tau^{-1}(\tau^{-1}(t))) - x(\tau^{-1}(\tau^{-1}(t)))]}{p(\tau^{-1}(t))p(\tau^{-1}(\tau^{-1}(t)))} \\ &\geq \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(t))p(\tau^{-1}(\tau^{-1}(t)))} z(\tau^{-1}(\tau^{-1}(t))). \end{aligned} \tag{5}$$

Now $\tau(t) \geq t$ and τ is strictly increasing, so $\tau^{-1}(t)$ is increasing and $t \geq \tau^{-1}(t)$. Thus,

$$\tau^{-1}(t) \geq \tau^{-1}(\tau^{-1}(t)),$$

and since $z'(t) > 0$, we have

$$z\left(\tau^{-1}(t)\right) \geq z\left(\tau^{-1}(\tau^{-1}(t))\right).$$

Substituting the last inequality into (5) yields

$$x(t) \geq \psi(t) z(\tau^{-1}(t)) \text{ for } t \geq t_{\lambda_0}. \tag{6}$$

Since $\lim_{t\to\infty} \varphi(t,\xi) = \infty$, we can choose $t_4 \geq t_{\lambda_0}$ such that $\varphi(t,\xi) \geq t_{\lambda_0}$ for all $t \geq t_4$. Thus, it follows from (6) that

$$x(\varphi(t,\xi)) \geq \psi(\varphi(t,\xi)) z\left(\tau^{-1}(\varphi(t,\xi))\right) \quad \text{for } t \geq t_4. \tag{7}$$

Using (7) in (1), we obtain

$$z^{(n)}(t) + \int_a^b q(t,\xi)\psi^\delta(\varphi(t,\xi)) z^\delta\left(\tau^{-1}(\varphi(t,\xi))\right) d\xi \leq 0. \tag{8}$$

Since τ and z are strictly increasing and φ is nonincreasing in ξ, from (8) we see that

$$z^{(n)}(t) + \left(\int_a^b q(t,\xi)\psi^\delta(\varphi(t,\xi)) d\xi\right) z^\delta\left(\tau^{-1}(\varphi(t,b))\right) \leq 0,$$

or

$$z^{(n)}(t) + Q(t) z^{\delta}\left(\tau^{-1}(\varphi_1(t))\right) \leq 0 \quad \text{for } t \geq t_4. \tag{9}$$

It follows from (4) and (9) that

$$z^{(n)}(t) + \frac{(\lambda_0)^{\delta}}{((n-1)!)^{\delta}} Q(t) \left(\tau^{-1}(\varphi_1(t))\right)^{\delta(n-1)} \left(z^{(n-1)}\left(\tau^{-1}(\varphi_1(t))\right)\right)^{\delta} \leq 0, \tag{10}$$

for $t \geq t_4$. Letting $y(t) = z^{(n-1)}(t)$, we see that $y(t)$ is a positive solution of the first-order delay differential inequality

$$y'(t) + \frac{(\lambda_0)^{\delta}}{((n-1)!)^{\delta}} Q(t) \left(\tau^{-1}(\varphi_1(t))\right)^{\delta(n-1)} y^{\delta}\left(\tau^{-1}(\varphi_1(t))\right) \leq 0, \tag{11}$$

for all $t \geq t_4$.

Integrating (11) from t to $u \geq t \geq t_4$ and letting $u \to \infty$, we see that

$$y(t) \geq \frac{(\lambda_0)^{\delta}}{((n-1)!)^{\delta}} \int_t^{\infty} Q(s) \left(\tau^{-1}(\varphi_1(s))\right)^{\delta(n-1)} y^{\delta}\left(\tau^{-1}(\varphi_1(s))\right) ds$$

for $t \geq t_4$. The function $y(t)$ is strictly decreasing on $[t_4, \infty)$, and so by [19, Theorem 1], there exists a positive solution of equation (3). This contradicts the fact that equation (3) is oscillatory and completes the proof. ∎

It is well known from [8] that if

$$\liminf_{t \to \infty} \int_{\eta(t)}^{t} R(s) ds > \frac{1}{e}, \tag{12}$$

then the first-order delay differential equation

$$x'(t) + R(t) x(\eta(t)) = 0$$

is oscillatory, where R, $\eta \in C([t_0, \infty), \mathbb{R})$ with $R(t) \geq 0$, $\eta(t) \leq t$, and $\lim_{t \to \infty} \eta(t) = \infty$.

Thus, from Theorem 1, we have the following result.

Corollary 1. *Let conditions (i)–(iv) hold and* $\delta = 1$. *Assume that* $\varphi(t, \xi) \leq \tau(t)$ *for* $(t, \xi) \in [t_0, \infty) \times [a, b]$. *If*

$$\liminf_{t \to \infty} \int_{\tau^{-1}(\varphi_1(t))}^{t} Q(s) \left(\tau^{-1}(\varphi_1(s))\right)^{n-1} ds > \frac{(n-1)!}{e}, \tag{13}$$

then equation (1) *is oscillatory.*

Proof. From (13), we can choose a positive constant λ_0 with $0 < \lambda_0 < 1$ such that equation (3) is oscillatory and

$$\liminf_{t\to\infty} \lambda_0 \int_{\tau^{-1}(\varphi_1(t))}^{t} Q(s) \left(\tau^{-1}(\varphi_1(s))\right)^{n-1} ds > \frac{(n-1)!}{e}.$$

Thus, by Theorem 1 and (12), the conclusion of Corollary 1 holds. ∎

THEOREM 2. *Let conditions (i)–(iv) hold and $\delta < 1$. Assume that $\varphi(t,\xi) \leq \tau(t)$ for $(t,\xi) \in [t_0,\infty) \times [a,b]$. If*

$$\int_{t_0}^{\infty} Q(s) \left(\tau^{-1}(\varphi_1(s))\right)^{\delta(n-1)} ds = \infty, \tag{14}$$

then equation (1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of equation (1), say $x(t) > 0$ for $t \in [t_1,\infty)$, $x(\varphi(t,\xi)) > 0$ for $(t,\xi) \in [t_1,\infty) \times [a,b]$, and $\psi(t) > 0$ for $t \geq t_1$. Proceeding as in the proof of Theorem 1, we again arrive at (11). Since y is decreasing and $\tau^{-1}(\varphi_1(t)) \leq t$, we have

$$y\left(\tau^{-1}(\varphi_1(t))\right) \geq y(t).$$

Using this in (11), we see that

$$y'(t) + \frac{\lambda_0}{((n-1)!)^{\delta}} Q(t) \left(\tau^{-1}(\varphi_1(t))\right)^{\delta(n-1)} y^{\delta}(t) \leq 0 \quad \text{for} \quad t \geq t_4. \tag{15}$$

An integration of (15) from t_4 to t gives

$$\int_{t_4}^{\infty} Q(s) \left(\tau^{-1}(\varphi_1(s))\right)^{\delta(n-1)} ds \leq \frac{((n-1)!)^{\delta}}{\lambda_0} \frac{y^{1-\delta}(t_4)}{1-\delta} < \infty$$

which contradicts (14) and completes the proof. ∎

Next, we present the following interesting result in which we need to assume that the function φ in condition (iv) is nondecreasing in t, i.e., φ is nondecreasing with respect to its first variable.

THEOREM 3. *Let conditions (i)–(iv) hold and assume that the function φ with $\varphi(t,\xi) \leq \tau(t)$ is nondecreasing in t. If $\delta = 1$ and*

$$\limsup_{t\to\infty} \int_{\tau^{-1}(\varphi_1(t))}^{t} Q(s) \left(\tau^{-1}(\varphi_1(s))\right)^{n-1} ds > (n-1)!\,, \tag{16}$$

or $\delta < 1$ and

$$\limsup_{t\to\infty} \int_{\tau^{-1}(\varphi_1(t))}^{t} Q(s) \left(\tau^{-1}(\varphi_1(s))\right)^{\delta(n-1)} ds = \infty, \tag{17}$$

then equation (1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of equation (1) with $x(t) > 0$ for $t \in [t_1, \infty)$, $x(\varphi(t,\xi)) > 0$ for $(t,\xi) \in [t_1,\infty) \times [a,b]$, and $\psi(t) > 0$ for $t \geq t_1$, for some $t_1 \geq t_0$.

We proceed as in the proof of Theorem 1 and fix $\lambda_0 \in (0,1)$ as we did there with the added stipulation that if $\delta = 1$, we also want

$$\limsup_{t\to\infty} \lambda_0 \int_{\tau^{-1}(\varphi_1(t))}^{t} Q(s) \left(\tau^{-1}(\varphi_1(s))\right)^{\delta(n-1)} ds > (n-1)! \tag{18}$$

to hold. Thus, we again arrive at an inequality of the form of (10) holding for $t \geq t_4$. Integrating (10) from $\tau^{-1}(\varphi_1(t))$ to t, we obtain

$$\begin{aligned} &z^{(n-1)}\left(\tau^{-1}(\varphi_1(t))\right) \geq \\ &\int_{\tau^{-1}(\varphi_1(t))}^{t} \frac{\lambda_0}{((n-1)!)^\delta} Q(s) \left(\tau^{-1}(\varphi_1(s))\right)^{\delta(n-1)} \left(z^{(n-1)}\left(\tau^{-1}(\varphi_1(s))\right)\right)^\delta ds \\ \geq\ & \left[\int_{\tau^{-1}(\varphi_1(t))}^{t} \frac{\lambda_0}{((n-1)!)^\delta} Q(s) \left(\tau^{-1}(\varphi_1(s))\right)^{\delta(n-1)} ds\right] \left(z^{(n-1)}\left(\tau^{-1}(\varphi_1(t))\right)\right)^\delta \end{aligned}$$

which can be written as

$$\left(z^{(n-1)}\left(\tau^{-1}\left(\varphi_1(t)\right)\right)\right)^{1-\delta} \geq \left[\int_{\tau^{-1}(\varphi_1(t))}^{t} \frac{\lambda_0}{((n-1)!)^\delta} Q(s) \left(\tau^{-1}(\varphi_1(s))\right)^{\delta(n-1)} ds\right]. \tag{19}$$

Now, if $\delta = 1$, then taking the lim sup of both sides of (19) as $t \to \infty$, we obtain

$$\limsup_{t\to\infty} \lambda_0 \int_{\tau^{-1}(\varphi_1(t))}^{t} Q(s) \left(\tau^{-1}(\varphi_1(s))\right)^{n-1} ds \leq (n-1)!\,, \tag{20}$$

which contradicts (18).

Next, if $\delta < 1$, then by the fact that $z^{(n-1)}(t)$ is positive and decreasing, there is a positive constant M such that

$$z^{(n-1)}(t) \leq M. \tag{21}$$

Using (21) in (19), and then taking the lim sup of both sides of (19) as $t \to \infty$, we see that

$$\limsup_{t\to\infty} \int_{\tau^{-1}(\varphi_1(t))}^{t} Q(s) \left(\tau^{-1}(\varphi_1(s))\right)^{\delta(n-1)} ds \leq K,$$

where $K = \frac{((n-1)!)^\delta M^{1-\delta}}{\lambda_0}$, which contradicts (17) and completes the proof. ∎

Next, we present an oscillation result in which we replace the condition on equation (3) by one that is quite easy to verify.

THEOREM 4. *Let conditions (i)–(iv) hold and assume that* $\varphi(t,\xi) \leq \tau(t)$ *for* $(t,\xi) \in [t_0,\infty) \times [a,b]$. *If*

$$\int_{t_0}^{\infty} Q(s)ds = \infty, \tag{22}$$

then equation (1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of equation (1), say $x(t) > 0$ for $t \in [t_1,\infty)$, $x(\varphi(t,\xi)) > 0$ for $(t,\xi) \in [t_1,\infty) \times [a,b]$, and $\psi(t) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Proceeding as in the proof of Theorem 1, we again arrive at (9). Since z is positive and increasing on $[t_4,\infty) \subseteq [t_{\lambda_0},\infty)$, there exist a $t_5 \in [t_4,\infty)$ and a positive constant c such that

$$z(t) \geq c \quad \text{for } t \geq t_5,$$

and so

$$z\left(\tau^{-1}(\varphi_1(t))\right) \geq c \quad \text{for } t \geq t_6, \tag{23}$$

where $\tau^{-1}(\varphi_1(t)) \geq t_5$ for $t \geq t_6$.

Using (23) in (9), we obtain

$$z^{(n)}(t) + c_1 Q(t) \leq 0 \quad \text{for } t \geq t_6, \tag{24}$$

where $c_1 = c^{\delta}$. Integrating (24) from t_6 to t and using then the fact that $z^{(n-1)}(t) > 0$, we get

$$\int_{t_6}^{t} Q(s)ds \leq \frac{1}{c_1} z^{(n-1)}(t_6) < \infty \quad \text{as} \quad t \to \infty,$$

which contradicts (22) and completes the proof. ∎

REMARK 1. *The results obtained here are also valid in the case when the function* φ *in condition (iv) is nondecreasing in* ξ. *In this case, we replace*

$$\varphi_1(t) := \varphi(t,b) \quad \textit{by} \quad \varphi_2(t) := \varphi(t,a).$$

We conclude this paper with the following examples and remarks to illustrate the above results. Our first example is concerned with the case where $p(t) \to \infty$ as $t \to \infty$, and in the second example p is a constant function.

Example 1. *Consider the even-order neutral differential equation*

$$(x(t)+4tx(2t))^{(n)}+\int_1^2 (t-5\xi)^{3/7}x^{3/7}\left(\frac{t}{5}-\xi\right)d\xi=0,\quad t\geq 12. \tag{25}$$

Here $p(t)=4t$, $\tau(t)=2t$, $a=1$, $b=2$, $q(t,\xi)=(t-5\xi)^{3/7}$, $\delta=3/7$, *and* $\varphi(t,\xi)=t/5-\xi$. *Then, it is easy to see that conditions (i)–(iv) hold and*

$$\varphi_1(t)=(t-10)/5,\quad \tau^{-1}(t)=t/2,\quad \tau^{-1}\left(\tau^{-1}(t)\right)=t/4,$$
$$\tau^{-1}\left(\varphi_1(t)\right)=(t-10)/10,\quad \psi(t)=\frac{1}{2t}\left(1-\frac{1}{t}\right)\geq\frac{11}{24t},$$
$$Q(t)\geq\int_1^2 (t-5\xi)^{3/7}\left(\frac{55}{24(t-5\xi)}\right)^{3/7}d\xi=\left(\frac{55}{24}\right)^{3/7},$$

Next, condition (14) *becomes*

$$\begin{aligned}\int_{t_0}^{\infty} Q(s)\left(\tau^{-1}(\varphi_1(s))\right)^{\delta(n-1)}ds &\geq \int_{12}^{\infty}\left(\frac{55}{24}\right)^{3/7}\left(\frac{s-10}{10}\right)^{3(n-1)/7}ds\\ &= \left(\frac{55}{24\times 10^{n-1}}\right)^{3/7}\int_{12}^{\infty}(s-10)^{3(n-1)/7}ds=\infty.\end{aligned}$$

Therefore, by Corollary 2, equation (25) *is oscillatory.*

Example 2. *Consider the fourth-order neutral differential equation*

$$(x(t)+5x(t+1))''''+\int_1^2\frac{k\xi}{t^4}x\left(\frac{t}{6}+\frac{1}{\xi}\right)d\xi=0,\quad t\geq 4. \tag{26}$$

Here $p(t)=5$, $\tau(t)=t+1$, $a=1$, $b=2$, $q(t,\xi)=k\xi/t^4$ *with* $k\geq 3060$, $\delta=1$, *and* $\varphi(t,\xi)=t/6+1/\xi$. *Then, it is easy to see that conditions (i)–(iv) hold,*

$$\varphi_1(t)=(t+3)/6,\quad \tau^{-1}(t)=t-1,\quad \tau^{-1}\left(\tau^{-1}(t)\right)=t-2,\quad \psi(t)=4/25,$$
$$\tau^{-1}\left(\varphi_1(t)\right)=(t-3)/6,\quad \textit{and}\ \ Q(t)=\frac{4}{25}\int_1^2\frac{k\xi}{t^4}d\xi=\frac{6k}{25t^4}.$$

Next, in view of (16), *we see that*

$$\begin{aligned}&\int_{\tau^{-1}(\varphi_1(t))}^{t} Q(s)\left(\tau^{-1}(\varphi_1(s))\right)^{n-1}ds\\ &=\int_{(t-3)/6}^{t}\frac{6k}{25s^4}\left(\frac{s-3}{6}\right)^3 ds\\ &=\frac{6k}{25\times 6^3}\int_{(t-3)/6}^{t}\left(\frac{1}{s}-\frac{9}{s^2}+\frac{27}{s^3}-\frac{27}{s^4}\right)ds\\ &=\frac{k}{900}\left(\ln\frac{6t}{t-3}+\frac{9}{t}-\frac{27}{2t^2}+\frac{9}{t^3}-\frac{54}{t-3}+\frac{486}{(t-3)^2}+\frac{1944}{(t-3)^3}\right).\end{aligned} \tag{27}$$

Taking the $\limsup$ *as* $t \to \infty$ *in* (27)*, we see that condition* (16) *holds, so by Theorem 3, equation* (26) *is oscillatory.*

REMARK 2. *The results of this paper can be easily extended to the even-order neutral differential equation*

$$\left(r(t)\left[(z^{(n-1)}(t)\right]^{\alpha}\right)' + \int_a^b q(t,\xi)x^{\delta}\left(\varphi(t,\xi)\right)d\xi = 0$$

under the two conditions

$$\int_{t_0}^{\infty} r^{-1/\alpha}(t)dt = \infty$$

and

$$\int_{t_0}^{\infty} r^{-1/\alpha}(t)dt < \infty,$$

where $r \in C\left([t_0,\infty),(0,\infty)\right)$*,* $z(t) = x(t) + p(t)x(\tau(t))$*,* $\delta \geq \alpha$ *with* α *is the quotient of odd positive integers, and the other functions and constant* δ *in the equation are defined as in this paper.*

REMARK 3. *It would be of interest to study the oscillatory behavior of all solutions of* (1) *for* $p(t) \leq -1$ *with* $p(t) \not\equiv -1$ *for large* t*.*

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