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ASYMPTOTIC BEHAVIOR OF FIRST ORDER SCALAR LINEAR AUTONOMOUS RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. This paper studies the asymptotic behavior of first order scalar linear autonomous Retarded Functional Differential Equations (RFDE). A spectral decomposition of the solution in terms of the exponential solutions is used to define the Dominant Spectral Component as the sum (with appropriate coefficients) of exponential solutions with characteristic roots of maximal real part, and likewise the Predominant Spectral Components as the sum of the exponential solutions with characteristic roots of maximal real part plus those (if any) with real parts greater than or equal to zero. Exponential bounds on the differences between the Solution and the Dominant Spectral Component (Predominant Spectral Components) provide a framework to investigate the exponential convergence to asymptotic behavior. Exponential solutions in the Dominant Spectral Component with real (complex) characteristic roots give rise to nonoscillatory (oscillatory) asymptotic behavior. Numerics for a simple RFDE illuminates the characteristics of the asymptotic behavior, illustrates how the asymptotic behavior depends on the characteristic roots in the right-most half plane, and demonstrates that the exponential convergence to asymptotic behavior is fairly rapid in a few delays. Tauberian techniques are used to express the asymptotic behavior of the integral of the solution in terms of its Laplace transform.

Key Words. retarded functional differential equations, asymptotic behavior, fundamental solution, characteristic function

AMS(MOS) subject classification. 34K06, 34A30

1. Introduction.

This paper studies the asymptotic behavior of first order scalar linear autonomous Retarded Functional Differential Equation (RFDE)

$$x'(t) = \int_0^a A(ds)x(t-s),\ t > 0, \tag{1}$$

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with initial condition $x(t) = h(t), -a \leq t \leq 0$. The Borel measure $A(ds)$ in the integral is of the form

$$A(ds) = \sum_{i=0}^{N} A_i \delta(s - \theta_i) + A(s)ds, \tag{2}$$

where θ_i , $A_i \in \Re$, $0 = \theta_0 < \theta_1 ... < \theta_N = a$, and $A(s)$ is integrable on $[0, a]$. Reference to the symbol "A" as Borel measure, real number, or function will be evident from the context. The terms $\sum_{i=0}^{N} A_i \delta(s - \theta_i)$ and $A(s)ds$ respectively give rise to the discrete delays and the distributed delay aspects of the RFDE.

A classic treatment of the asymptotic behavior of the Solution $x(t)$ of Differential-Difference Equations (RFDE with discrete delays) is found in Reference [3]; in particular in Chapter 4 for the Solution $x(t)$, and in Chapter 7 for the Tauberian approach for the integral of the solution. Further developments are to be found in References [10],[11],[12], and [33]. The emphasis in these papers is on small/harmless delays in the sense that the RFDE shares a similar nonoscillatory asymptotic behavior as the corresponding delay-less ODE. Reference [13] is in the same spirit as this paper it that it exhibits graphical plots to compare the solution and its asymptotic exponential approximation of the RFDE $x'(t) = bx(t-1)$ with initial condition $h(t) = 1$ for $t \in [0, 1]$, observes that the approximation is surprisingly close even for small t, and provides an analytical explanation of the observation. References [17] and [18] follow a spectral approach for the case of a single dominant root of the characteristics function for a broader class of Functional Differential Equations. Some of the general references to the RFDE subject matter are [1], [2], [7], [8], [14], [20], and [32].

This paper extends the treatment of asymptotic behavior by:

1. Considering a general RFDE that incorporates both discrete and distributed delays.
2. Using a spectral decomposition of the Solution $x(t)$ in terms of the exponential solutions to define the Dominant Spectral Component $y_0(t)$ as the sum (with appropriate coefficients) of exponential solutions with characteristic roots of maximal real part, and likewise the Predominant Spectral Components $y_1(t)$ as sum (with appropriate coefficients) of the exponential solutions with characteristic roots of maximal real part plus those (if any) with real part greater than or equal to zero.
3. Exploiting exponential bounds on the differences $x(t) - y_0(t)$ and $x(t) - y_1(t)$ to establish exponential convergence to asymptotic behavior in relation to the Dominant Spectral Component $y_0(t)$ and

the Predominant Spectral Components $y_1(t)$.

4. Investigating the case where the leading characteristic roots are complex giving rise to oscillatory asymptotic behavior, in addition to the case where the leading characteristics root(s) are real giving rise to nonoscillatory asymptotic behavior.
5. Utilizing Numerics for a simple RFDE to elaborate on the characteristics of the asymptotic behavior, to illustrate how the asymptotic behavior depends on the characteristic roots in the right-most half plane, and to demonstrate that the exponential convergence to asymptotic behavior is fairly rapid in a few delays.
6. Applying Tauberian treatment to obtain results on the asymptotic behavior of the integral of the Solution $x(t)$ in terms of its Laplace transform.

The flow in this paper follows the conventional path of stipulating the functional differential equations to be studied, conducting mathematical analysis, exploring numerics, observing the phenomenon that asymptotic behavior is attained quickly after a few delays (also see Reference [13]), and providing an explanation of this phenomenon at the end of the document. This observation of rapid convergence to asymptotic behavior is striking because although strictly speaking asymptotic behavior is limiting behavior, it is usually construed as large time behavior. In reality, the research activity and subsequent mathematical analysis for this paper was motivated by a search for an explanation of the phenomenon of rapid convergence to asymptotic behavior. This starting point was made possible by access on a desktop to hardware and software for the numerical solutions of functional differential equations. The difficulty of this hardware and software access from a desktop was essentially impossible up to 50 years ago, and a formidable challenge over the next 20 years. However, today it is a routine matter.

Consider the operator

$$L(x(t)) = \int_0^a A(ds)x(t-s). \tag{3}$$

We have

$$L(e^{\lambda t}) = \Delta(\lambda)e^{\lambda t}, \tag{4}$$

where the Characteristic Function

$$\Delta(\lambda) = \lambda - \int_0^a A(ds)e^{-\lambda s}. \tag{5}$$

Now suppose that $\Delta(\lambda)$ has roots λ_r with multiplicity m_r. We have that the exponential solutions $\{t^j e^{\lambda t}\}$ satisfy Equation (1) excluding the initial condition, since $L(t^j e^{\lambda t}) = 0$ for $j = 0, ...m_r - 1$. Furthermore, any arbitrary sum of exponential solutions with coefficients $\{a_{rj}\}$ is a solution of Equation (1) excluding the initial condition.

Based on the completeness and independence of the exponential solutions established in Reference [26], the solution of Equation (1) is given by

$$x(t) = \sum_r \sum_{j=0}^{m_r-1} a_{rj} t^j e^{\lambda_r t}, \tag{6}$$

where the coefficients $\{a_{rj}\}$ depend on the Characteristic Function $\Delta(z)$ and the initial function h.

We order the roots $\{\lambda_r\}$ starting at $r = 1$ by decreasing value of $\Re(\lambda_r)$ and identify the leading roots with maximum real part. We define the Dominant Spectral Component $y_0(t)$ as the sum of the exponential solutions with maximum real part. Set $\gamma_0 = \max\{\Re(\lambda_r)\}$ for λ_r not a leading root. The expression

$$x(t) = y_0(t) + O(e^{(\gamma_0+\epsilon)t}) \tag{7}$$

for arbitrary $\epsilon > 0$ provides the basis for investigating the asymptotic behavior in relation to the Dominant Spectral Component. We also define the Predominant Spectral Components $y_1(t)$ as the sum of the exponential solutions with maximum real part plus those exponential solutions with $\Re(\lambda_r) \geq 0$, if any. Set $\gamma_1 = \max\{\Re(\lambda_r)\}$ for λ_r not a member of the characteristic roots comprising the Predominant Spectral Components. The expression

$$x(t) = y_1(t) + O(e^{(\gamma_1+\epsilon)t}) \tag{8}$$

for arbitrary $\epsilon > 0$ provides the basis for investigating the asymptotic behavior in relation to the Predominant Spectral Components. Note that the Dominant Spectral Component $y_0(t)$ and the Predominant Spectral Components $y_1(t)$ are the same when (i) all the characteristic roots lie in the left half plane, or (ii) all the characteristic roots except the leading roots lie in the left half plane. Exploiting the expressions in (7) and (8) requires information on the location of the characteristic roots $\{\lambda_r\}$ in the rightmost half of the complex plane. In Section 5 we consider the simple RFDE $x'(t) = A_0 x(t) + A_1 x(t-1)$ for which this information is readily available.

Note that while an infinite (large) number of exponential functions is required for a fit to the initial data $h(t)$, only a small number of exponential solutions are involved in the asymptotic behavior. Numerics presented in Section 5 demonstrates that the transition period to asymptotic behavior is quick, after a few delays, as is consistent with exponential convergence.

The nature of the limit of the Laplace transform $\hat{X}(z)$ as $z \to 0$ is the essence of the Tauberian approach in extracting information on the asymptotic behavior of the integral of $x(t)$ over the interval $[0, T]$ as $T \to \infty$. For example if the Fundamental Solution $\Phi(t)$ is integrable on $[0, \infty)$ we have

$$\hat{\Phi}(z) = \int_0^\infty \Phi(t) e^{-zt} dt = 1/\Delta(z) \tag{9}$$

and from the limit $z \to 0$ we have

$$\int_0^\infty \Phi(t) dt = 1/\{-\int_0^a A(ds)\}. \tag{10}$$

Issues arise if the Characteristic Function $\Delta(z)$ has a root at $z = 0$ so that

$$\int_0^a A(ds) = 0. \tag{11}$$

This is explored further in Section 4.

The toolkit used in this paper to obtain results on the asymptotic behavior of autonomous RFDE is based on the Characteristic Function $\Delta(z)$ and its associated Fundamental Solution $\Phi(t)$ and exponential solutions $\{t^j e^{\lambda_r t}\}$. Note should be made of an alternative toolkit in the paper [9] (where other references are cited) for a first order scalar nonautonomous RFDE with boundary condition based on the formulation of a RFDE using a Volterra operator. We have a first order scalar linear nonautonomous RFDE

$$(Mx)(t) \equiv x'(t) + (Bx)(t) = f(t),\ t \in [0, \omega], \tag{12}$$

with boundary condition

$$lx = c, \tag{13}$$

where B is a linear continuous Volterra operator, and l is a linear bounded functional. The function spaces associated with B and l are described in Reference [9]. The interested reader should consult the books in References [2] and [1].

The autonomous RFDE considered in this paper is a particular case of equations (12) and (13), with

$$(Bx)(t) = -\int_0^a A(ds)x(t-s),\ t>0, \tag{14}$$

so that

$$(Mx)(t) \equiv x'(t) = \int_0^a A(ds)x(t-s) + f(t),\ t>0, \tag{15}$$

with initial condition

$$lx = x(0) = c, \tag{16}$$

where B is a Volterra operator of convolution type with measure kernel $A(ds)$ with compact support on $[0, a]$, and l corresponds to an initial condition. Note that the initial data $h(t)$ for $-a \le t < 0$ can be incorporated within the forcing function $f(t)$.

The Cauchy Function $C(t, s)$ for equation (12) satisfies the equation with $x(\xi) = 0$ for $\xi < s$ and $C(s, s) = 1$. The Cauchy Function for the autonomous equation (15) is given in explicit form in terms of the Fundamental Solution $\Phi(t)$ in (22) as

$$C(t, s) = \Phi(t-s). \tag{17}$$

The Volterra operator toolkit contains maximal principles that transform differential equalities to solution inequalities, yielding results on positivity of the Cauchy Function, nonoscillation, and stability. These results have direct analogs for the autonomous case with initial condition, in which the Fundamental Solution plays the role of the Cauchy Function.

There is an immense literature on the applications of general functional differential equations. The interested reader is referred to the books in References [24] and [25] for a comprehensive coverage. In particular, there is a broad range of applications of first order functional differential equations. Examples that are a part of the everyday human experience are: temperature variations encountered by a person taking a shower in Reference [24] (pages 6-7), dynamics of the business cycle in References [19] and [23], spread of infectious diseases in Reference [5], and modeling of the formation of blood cells in Reference [4].

It is expected that the results on asymptotic behavior for first order scalar RFDE will generalize to a n-vector RFDE albeit with heavier mathematical machinery. The asymptotic behavior of n-vector RFDE is expected to have a wider repertoire, such as the existence of nontrivial small solutions that decrease faster than any exponential.

The remainder of the paper proceeds as follows: Section 2 on Preliminaries sets up the framework to study asymptotic behavior. The main results of the paper are in Sections 3 and 4 respectively on asymptotic behavior of the solution and the integral of the solution. Finally, Section 5 on Numerics gives a numerical study illustrating the full range of asymptotic behavior for a simple RFDE.

2. Preliminaries. We consider the first order linear autonomous RFDE

$$x'(t) = \int_0^a A(ds)x(t-s) + f(t),\ t > 0, \tag{18}$$

with initial condition $x(t) = h(t)$ for $-a \le t \le 0$, and forcing function $f(t) \in L_1^{loc}(0,\infty)$, where $A(ds)$ is a Borel measure on the real line $\mathbb{R}$ with support on the interval [0,a].

An alternative formulation is that $A(s)$ is a function of bounded variation on the interval [0,a] or that the RFDE is expressed in terms of distributions. However, we will consider the equivalent Borel measure formulation.

From the Lebesgue-Radon-Nikodym decomposition theorem (see Chapter 5 and Theorem 19.61 of Reference [21]) a Borel measure $A(ds)$ can be delineated into three parts:

$$A(ds) = A(s)ds + \sum_{i=0}^{\infty} A_i\delta(s-\theta_i) + A_s(ds), \tag{19}$$

where

1. $A(s)ds$ represents the absolutely continuous part,
2. $\sum_{i=0}^{\infty} A_i\delta(s-\theta_i)$ represents the discrete singular part with an (possible) infinite number of point masses (or atoms) at the points θ_i, and
3. $A_s(ds)$ represents the continuous singular part.

We shall restrict the Borel measure $A(ds)$ to have a finite number of discrete atoms at $0 = \theta_0 < \theta_1 ... < \theta_N = a$ in the discrete singular component, and a zero continuous singular component.

We make Assumption A, that either

1. $A_N \neq 0$, i.e. $A(ds)$ has a non-zero atom at a, or
2. $\exists\, \delta > 0$ such that $A(s) \ge 0$ (or $A(s) \le 0$) for $s \in (a-\delta, a)$ and A(s) is not the null function on $(a-\delta/2, a)$.

As the initial data is a function $h(t)$, $-a \leq t \leq 0$, it is convenient to consider the problem in some suitable function space. We consider the function spaces F_1 and F_2:

F_1 $h \in M^2[-a,0]$, h is Lebesgue measurable on $[-a,0]$, $h(0)$ is well defined, $\int_{-a}^{0} |h(t)|^2 dt < \infty$, and $||h||_1 = \{|h(0)|^2 + \int_{-a}^{0} |h(t)|^2 dt\}^{1/2}$.

F_2 $h \in C[-a,0]$, $||h||_2 = \sup_{-a \leq t \leq 0} |h(t)|$.

Results on the existence, uniqueness and continuous dependence on the initial data can be found in Reference [6] for function space F_1 and Reference [20] for function space F_2.

THEOREM 1 (Reference [6]). *The representation of the Solution $x(t)$ for the RFDE in Equation (18) is given in the form*

$$x(t) = \Phi(t)h(0) + \int_{-a}^{0} d\alpha\, h(\alpha) \int_{-\alpha}^{a} A(ds)\Phi(t-s-\alpha) + \int_{0}^{t} \Phi(s) f(t-s) ds, \tag{20}$$

where the Fundamental Solution $\Phi(t)$ satisfies RFDE with initial data $h(0) = 1$ and $h(t) = 0$ for $t \in [-a, 0)$.

Note that $\Phi(t)$ satisfies a Volterra integro-differential equation

$$\Phi'(t) = \int_{0}^{t} A(ds)\Phi(t-s), \tag{21}$$

with initial condition $\Phi(0) = 1$, $\Phi(t) = 0$ for $t < 0$. Note also that this initial data is a member of the function space F_1.

THEOREM 2 (Reference [30]). *The Fundamental Solution $\Phi(t)$ for Equation (18) is given by*

$$\Phi(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{0}^{t} A_n(ds)(t-s)^n \tag{22}$$

where the measures $A_n(ds)$ are defined as follows:

1. $A_0(ds) = \delta(s)$,
2. $A_1(ds) = A(ds)$, *and*
3. $A_n(ds) = A_{n-1}(ds) * A_1(ds)$ *for* $n \geq 2$,

where $*$ denotes the convolution of two measures.

Note that the calculation for the Fundamental Solution $\Phi(t)$ is simplified by the restriction of a finite number of discrete atoms in the discrete singular component, and a zero continuous singular component.

We consider the homogeneous equation:

$$x'(t) = \int_0^a A(ds)x(t-s),\ t > 0, \tag{23}$$

with initial condition $x(t) = h(t), -a \leq t \leq 0$.

The Laplace transform $\mathcal{L}(x(t))(z) = \hat{X}(z)$ of $x(t)$ is defined by

$$\hat{X}(z) = \int_0^\infty x(t)e^{-zt}dt. \tag{24}$$

Taking the Laplace transform of both sides of Equation (23), after some manipulation we obtain

$$\{z - \int_0^a A(ds)e^{-zs}\}\hat{X}(z) = \{h(0) + \int_{-a}^0 d\alpha\, h(\alpha) \int_{-\alpha}^a A(ds)\, e^{-z(\alpha+s)}\} \tag{25}$$

so that

$$\hat{X}(z) = B(z,h)/\Delta(z) \tag{26}$$

where the analytic function of the initial data h

$$B(z,h) = h(0) + \int_{-a}^0 d\alpha\, h(\alpha) \int_{-\alpha}^a A(ds)\, e^{-z(\alpha+s)}, \tag{27}$$

and the Characteristic Function

$$\Delta(z) = z - \int_0^a A(ds)e^{-zs}. \tag{28}$$

$B(z,h)$ is related to the bilinear form $\langle g,h\rangle$ for $g,h \in F_1$ defined as (see Reference [7])

$$\langle g,h\rangle = g(0)h(0) + \int_{-a}^0 d\alpha\, h(\alpha) \int_{-\alpha}^a A(ds)\, g(-\alpha - s) \tag{29}$$

as

$$B(z,h) = \langle e^{z(\cdot)}, h\rangle. \tag{30}$$

An exponential solution $e^{\lambda t}$ of the Equation (23) satisfies

$$\Delta(\lambda) = \lambda - \int_0^a A(ds)e^{-\lambda s} = 0. \tag{31}$$

Let $\{\lambda_r\}$ be the roots of $\Delta(\lambda) = 0$ arranged by decreasing value of $\Re(\lambda_r)$ starting at $r = 1$ and let m_r be the multiplicity of the root λ_r. We have the exponential solutions $\{t^j e^{\lambda_r t}\}$ for $t > 0$ and $j = 0 ... m_r - 1$ with associated initial data $\{t^j e^{\lambda_r t}\}$ for $-a \leq t \leq 0$.

The question arises as to whether a general solution $x(t)$ of Equation (23) can be written as a linear combination of exponential solutions in the form

$$x(t) = \sum_r \sum_{j=0}^{m_r - 1} a_{rj} t^j e^{\lambda_r t} \tag{32}$$

for $t > 0$.

It is well known (see Reference [26]) that the exponential functions $\{t^j e^{\lambda_r t}\}$ for $-a \leq t \leq 0$ and $j = 0 ... m_r - 1$ are complete and independent in the function spaces F_1 and F_2. An initial data function h can be written in the form

$$h(t) = \sum_r \sum_{j=0}^{m_r - 1} a_{rj} t^j e^{\lambda_r t}, -a \leq t \leq 0. \tag{33}$$

Reference [26] provided an approach to express the coefficients $\{a_{rj}\}$ in terms of $B(z, h)$ and $\Delta(z)$ and derivatives with respect to z. However, the approach involved an iteration method for solving for the coefficients $\{a_{rj}\}$.

In the paper we consider an alternative approach that pushes the iteration upstream so that explicit formulae are given for the coefficients $\{a_{rj}\}$. The approach is based on residue calculus of the Solution $x(t)$ expressed in the form of Laplace inverse as a contour integral in the z-plane

$$x(t) = \frac{1}{2\pi i} \int_{(c)} e^{zt} B(z, h) \Delta^{-1}(z) dz. \tag{34}$$

To obtain the contribution to the integral of a root λ_r of $\Delta(\lambda)$ with multiplicity m_r, we consider the integral over a small circle centered at λ_r containing no other roots of $\Delta(\lambda)$.

We have

$$e^{zt} = e^{\lambda_r t} e^{(z-\lambda_r)t} = e^{\lambda_r t} \sum_{n=0} \frac{t^n}{n!}(z-\lambda_r)^n, \tag{35}$$

$$B(z,h) = \sum_{n=0} \frac{1}{n!} \frac{\partial^n B}{\partial z^n}(\lambda_r, h)\,(z-\lambda_r)^n, \tag{36}$$

$$\Delta(z) = (z-\lambda_r)^{m_r} \sum_{n=0} \frac{1}{(n+m_r)!} \Delta^{(n+m_r)}(\lambda_r)(z-\lambda_r)^n. \tag{37}$$

We have

$$\Delta^{-1}(z) = (z-\lambda_r)^{-m_r} \sum_{n=0} D_n(\lambda_r)(z-\lambda_r)^n \tag{38}$$

where $D_0(\lambda_r) = m_r!/\Delta^{(m_r)}(\lambda_r)$ and $D_n(\lambda_r)$ for $n \geq 1$ is obtained recursively from

$$\sum_{m=0}^{n} \frac{1}{(m+m_r)!} D_{n-m}(\lambda_r)\Delta^{(m+m_r)}(\lambda_r) = 0. \tag{39}$$

We have

$$e^{zt}B(z,h)\Delta^{-1}(z) = e^{\lambda_r t}(z-\lambda_r)^{-m_r} \sum_{k_1+k_2+k_3=0} \frac{t^{k_1}}{k_1!k_2!} \frac{\partial^{k_2} B}{\partial z^{k_2}}(\lambda_r, h)\, D_{k_3}(\lambda_r)(z-\lambda_r)^{k_1+k_2+k_3}. \tag{40}$$

The residue for the root λ_r is the coefficient of $(z-\lambda_r)^{-1}$ corresponding to $k = k_1 + k_2 + k_3 = m_r - 1$. The contribution to the Solution $x(t)$ arising from the root λ_r is given by

$$e^{\lambda_r t} \sum_{k_1+k_2+k_3=m_r-1} \frac{t^{k_1}}{k_1!k_2!} \frac{\partial^{k_2} B}{\partial z^{k_2}}(\lambda_r, h)\, D_{k_3}(\lambda_r)(z-\lambda_r)^{k_1+k_2+k_3}. \tag{41}$$

The coefficient a_{rj} is given by

$$a_{rj} = \frac{1}{j!} \sum_{k_2+k_3=m_r-1-j} \frac{1}{k_2!} \frac{\partial^{k_2} B}{\partial z^{k_2}}(\lambda_r, h)\, D_{k_3}(\lambda_r). \tag{42}$$

For example $a_{r,m_r-1} = B(\lambda_r, h)m_r/\Delta^{(m_r)}(\lambda_r)$, and for $m_r = 1$ $a_{r,0} = B(\lambda_r, h)/\Delta'(\lambda_r)$. Note that the evaluation of the coefficients a_{rj} only requires the computation of the first $m_r - 1$ values of D_n.

Let $x(t)$ be the solution of Equation (23) with initial data h. Define a functional element x_t by $x_t(\theta) = x(t+\theta)$ for $-a \leq \theta \leq 0$. We have $x_t \in C[-a,0]$ for $h \in C[-a,0]$ and $(x(t), x_t) \in M^2[-a,0]$ for $(h(0),h) \in M^2[-a,0]$. We define the solution operators $S(t)$ and $T(t)$ on $M^2[-a,0]$ and $C[-a,0]$ respectively as $S(t)h = (x(t), x_t)$ and $T(t)h = x_t$. $S(t)$ and $T(t)$ are strongly continuous semigroups of bounded operators on their respective function spaces. Further information is available in References [27] and [20] Chapter 7.

Let $\mathcal{N}$ be the set of indices $\{1, 2, ...N\}$ such that $\Re(\lambda_N) > \Re(\lambda_{N+1})$, and let $\gamma = \Re(\lambda_{N+1})$. For $h \in F_1$,

$$h = \sum_r \sum_{j=0}^{m_r-1} a_{rj} t^j e^{\lambda_r t} \tag{43}$$

we define

$$\bar{h} = \sum_{r\in\mathcal{N}} \sum_{j=0}^{m_r-1} a_{rj} t^j e^{\lambda_r t}, \ \bar{h} \in \bar{F}_1, \tag{44}$$

and

$$\tilde{h} = \sum_{r\notin\mathcal{N}} \sum_{j=0}^{m_r-1} a_{rj} t^j e^{\lambda_r t}, \ \tilde{h} \in \tilde{F}_1. \tag{45}$$

We have $h = \bar{h} + \tilde{h}$, F_1 is decomposed into the principal subspace $\bar{F}_1$ and the complementary subspace $\tilde{F}_1$, and $F_1 = \bar{F}_1 \bigotimes \tilde{F}_1$. Furthermore the subspaces $\bar{F}_1$ and $\tilde{F}_1$ are invariant under the action of the solution operator $S(t)$ so that $\bar{x}_t \in S(t)\bar{F}_1 \subset \bar{F}_1$ and $\tilde{x}_t \in S(t)\tilde{F}_1 \subset \tilde{F}_1$. The same applies to the function space F_2 with solution operator $T(t)$.

Correspondingly $x_t = \bar{x}_t + \tilde{x}_t$, and $x(t) = \bar{x}(t) + \tilde{x}(t)$, where for $t > 0$

$$\bar{x}(t) = \sum_{r\in\mathcal{N}} \sum_{j=0}^{m_r-1} a_{rj} t^j e^{\lambda_r t} \tag{46}$$

and

$$\tilde{x}(t) = \sum_{r\notin\mathcal{N}} \sum_{j=0}^{m_r-1} a_{rj} t^j e^{\lambda_r t}. \tag{47}$$

THEOREM 3 (References [27] and [20] page 214.). *Let $x(t)$ be a solution of Equation* (23) *and let $\epsilon > 0$. There exists $K(\epsilon) > 0$ such that*

$$||\tilde{x}_t)||_1 = ||S(t)\tilde{h}||_1 \leq K(\epsilon)e^{(\gamma+\epsilon)t}||\tilde{h}||_1 \tag{48}$$

and

$$||\tilde{x}_t)||_2 = ||T(t)\tilde{h}||_2 \leq K(\epsilon)e^{(\gamma+\epsilon)t}||\tilde{h}||_2. \tag{49}$$

Note that Equations (48) and (49) of Theorem 2.3 give an exponential bound $Ke^{(\gamma+\epsilon)t}$ on the portion $\tilde{x}(t)$ of the Solution $x(t)$ involving the exponential solutions with $\Re(\lambda_r) \leq \gamma$ and excluding those exponential solutions with $\Re(\lambda_r) > \gamma$. This provides the framework for investigating the asymptotic behavior of the Solution $x(t)$.

3. Asymptotic Behavior of Solution.

THEOREM 4. *Let $x(t)$ be a solution of Equation* (23)*. For $\epsilon > 0$*

$$x(t) = \sum_{r\in\mathcal{N}} \sum_{j=0}^{m_r-1} a_{rj}t^j e^{\lambda_r t} + O(e^{(\gamma+\epsilon)t}). \tag{50}$$

Proof. From the Theorem 2.3 we have

$$|\tilde{x}(t)| \leq K(\epsilon)e^{(\gamma+\epsilon)t}||\tilde{h}||_1 \tag{51}$$

as well as

$$|\tilde{x}(t)| \leq K(\epsilon)e^{(\gamma+\epsilon)t}||\tilde{h}||_2. \tag{52}$$

Hence result. □

Define

1. Leading root λ_r as the root(s) with the largest real part,
2. Dominant Spectral Component $y_0(t)$ to be the sum of the exponential solutions with the leading root(s), and
3. Predominant Spectral Components $y_1(t)$ to be the sum of the exponential solutions with the leading root(s) plus the exponential solutions with characteristic roots $\Re(\lambda_r) \geq 0$, if any.

We set $\mathcal{N}_0$ as the set of indices of the leading roots and $\gamma_0 = \max_{r\notin\mathcal{N}_0} \Re(\lambda_r)$. Likewise set $\mathcal{N}_1$ as the set of indices of the leading characteristic roots plus the characteristic roots with $\Re(\lambda_r) \geq 0$ (if any), and $\gamma_1 = \max_{r\notin\mathcal{N}_1} \Re(\lambda_r)$.

Note that by definition $\gamma_1 < 0$. The Dominant Spectral Component $y_0(t)$ is given by

$$y_0(t) = \sum_{r \in \mathcal{N}_0} \sum_{j=0}^{m_r - 1} a_{rj} t^j e^{\lambda_r t}. \tag{53}$$

The Predominant Spectral Components $y_1(t)$ is given by

$$y_1(t) = \sum_{r \in \mathcal{N}_1} \sum_{j=0}^{m_r - 1} a_{rj} t^j e^{\lambda_r t}. \tag{54}$$

The absolute difference between the Solution $x(t)$ and the Dominant Spectral Component $y_0(t)$ is

$$x(t) - y_0(t) = O(e^{(\gamma_0 + \epsilon)t}). \tag{55}$$

The absolute difference between the Solution $x(t)$ and the Predominant Spectral Components $y_1(t)$ is

$$x(t) - y_1(t) = O(e^{(\gamma_1 + \epsilon)t}). \tag{56}$$

Theorem 5.

$$\lim_{t \to \infty} \{x(t) - y_1(t)\} = 0. \tag{57}$$

If $\gamma_0 < 0$

$$\lim_{t \to \infty} \{x(t) - y_0(t)\} = 0. \tag{58}$$

Proof. Follows directly from Equations (56) and (55). ∎

If $\gamma_0 < 0$, such as when all the characteristic roots λ_r lie in the left half plane $\Re(z) < 0$, we have exponential convergence of the absolute difference. For μ_1 the real part of the leading roots, each with multiplicity $m_r = 1$, the relative difference between the Solution $x(t)$ and the Dominant Spectral Component $y_0(t)$ can be constructed as

$$e^{-\mu_1 t}\{x(t) - y_0(t)\} = O(e^{(\gamma_0 + \epsilon - \mu_1)t}). \tag{59}$$

and this converges exponentially with the appropriate choice of $\epsilon > 0$. Hence in general we have exponential convergence of the relative difference, but not necessarily exponential convergence of the absolute difference.

Let us consider the case of single leading real root λ_1 with multiplicity 1. The Dominant Spectral Component $y_0(t)$ is given by

$$y_0(t) = e^{\lambda_1 t} L(\lambda_1, h) \tag{60}$$

where

$$L(\lambda_1, h) = \frac{B(\lambda_1, h)}{\Delta'(\lambda_1)} = \frac{\{h(0) + \int_{-a}^{0} d\alpha\, h(\alpha) e^{-\lambda_1 \alpha} \int_{-\alpha}^{a} A(ds)\, e^{-\lambda_1 s}\}}{\{1 + \int_{0}^{a} A(ds)\, s e^{-\lambda_1 s}\}}. \tag{61}$$

Theorem 6. *Let the roots λ_r of $\Delta(\lambda) = 0$ have a single leading real root λ_1 with multiplicity 1. Then*

$$\lim_{t\to\infty} e^{-\lambda_1 t} x(t) = L(\lambda_1, h). \tag{62}$$

In particular

$$\lim_{t\to\infty} e^{-\lambda_1 t} \Phi(t) = 1/\{1 + \int_{0}^{a} A(ds)\, s e^{-\lambda_1 s}\}. \tag{63}$$

Proof.

$$x(t) = e^{\lambda_1 t} L(\lambda_1, h) + O(e^{(\gamma_0 + \epsilon)t}). \tag{64}$$

Choosing ϵ such that $\gamma_0 + \epsilon - \lambda_1 < 0$ we have

$$\lim_{t\to\infty} e^{-\lambda_1 t} x(t) = L(\lambda_1, h). \tag{65}$$

□

This result is well known for small/harmless delays in which there is an appropriate bound on the delay a. See References [10], [11], and [12].

Theorem 7. *Let the roots λ_r of $\Delta(\lambda) = 0$ have a single leading real root λ_1 with multiplicity 1. Then*

$$\lim_{t\to\infty} e^{-\lambda_1 t} \{x(t) - \Phi(t) B(\lambda_1, h)\} = 0. \tag{66}$$

If $\gamma_0 < 0$

$$\lim_{t\to\infty} \{x(t) - \Phi(t) B(\lambda_1, h)\} = 0. \tag{67}$$

Proof. From the representation of solutions

$$x(t) = \Phi(t)h(0) + \int_{-a}^{0} d\alpha\, h(\alpha) \int_{-\alpha}^{a} A(ds)\Phi(t-s-\alpha) \tag{68}$$

$$= \Phi(t)\{h(0) + \int_{-a}^{0} d\alpha\, h(\alpha) \int_{-\alpha}^{a} A(ds)\, e^{-\lambda_1(\alpha+s)}\} + O((\gamma_0+\epsilon)t) \tag{69}$$

$$= \Phi(t)B(\lambda_1, h) + O((\gamma_0+\epsilon)t). \tag{70}$$

since

$$\Phi(t) = e^{\lambda_1 t}/\Delta'(\lambda_1) + O((\gamma_0+\epsilon)t) \tag{71}$$

and

$$\Phi(t-s-\alpha) = \Phi(t)e^{-\lambda_1(s+\alpha)} + O((\gamma_0+\epsilon)t). \tag{72}$$

The results now follow. ∎

We note that $\lambda = 0$ is a root of $\Delta(\lambda)$ iff $\int_0^a A(ds) = 0$. In this case, the RFDE has constant solutions. There are instances in which the 0-root may not be a leading characteristic root. But if it is the sole leading zero, we have that all solutions are bounded with finite limit. The following corollary follows by putting $\lambda_1 = 0$.

COROLLARY 1. *Let $\lambda = 0$ be the sole leading root of $\Delta(\lambda)$ with multiplicity 1 so that $\int_0^a A(ds) = 0$ and $1 + \int_0^a A(ds)s \neq 0$. Then*

$$\lim_{t\to\infty} x(t) = L(0, h). \tag{73}$$

Let us consider the case of leading complex roots $\lambda_1 = \mu_1 + i\nu_1$ and $\lambda_2 = \mu_1 - i\nu_1$ each with multiplicity 1. We have

$$B(\lambda_1, h) = X_1(\mu_1,\nu_1,h) - iY_1(\mu_1,\nu_1,h);\; B(\lambda_2, h) = X_1(\mu_1,\nu_1,h) + iY_1(\mu_1,\nu_1,h), \tag{74}$$

and

$$\Delta'(\lambda_1) = X_0(\mu_1,\nu_1) - iY_1(\mu_1,\nu_1);\; \Delta'(\lambda_2) = X_0(\mu_1,\nu_1) + iY_1(\mu_1,\nu_1), \tag{75}$$

where

$$X_0(\mu_1, \nu_1) = 1 + \int_0^a A(ds) s e^{-\mu_1 s} \cos(\nu_1 s), \tag{76}$$

$$Y_0(\mu_1, \nu_1) = \int_0^a A(ds) s e^{-\mu_1 s} \sin(\nu_1 s), \tag{77}$$

$$X_1(\mu_1, \nu_1, h) = h(0) + \int_{-a}^0 d\alpha\, h(\alpha) \int_{-\alpha}^a A(ds)\, e^{-\mu_1(\alpha+s)} \cos(\nu_1(\alpha + s)), \tag{78}$$

$$Y_1(\mu_1, \nu_1, h) = \int_{-a}^0 d\alpha\, h(\alpha) \int_{-\alpha}^a A(ds)\, e^{-\mu_1(\alpha+s)} \sin(\nu_1(\alpha + s)). \tag{79}$$

The (complex) spectral components are

$$e^{\lambda_1 t} B(\lambda_1, h)/\Delta'(\lambda_1) = e^{\mu_1 t}(\cos(\nu_1 t) + i \sin(\nu_1 t))(X_1 - iY_1)/(X_0 - iY_0) \tag{80}$$

and

$$e^{\lambda_2 t} B(\lambda_2, h)/\Delta'(\lambda_2) = e^{\mu_1 t}(\cos(\nu_1 t) - i \sin(\nu_1 t))(X_1 + iY_1)/(X_0 + iY_0). \tag{81}$$

The Dominant Spectral Component $y_0(t)$ is given by

$$y_0(t) = 2e^{\mu_1 t}\{(X_1 X_0 + Y_1 Y_0) \cos(\nu_1 t) + (Y_1 X_0 - X_1 Y_0) \sin(\nu_1 t)\}/\{X_0^2 + Y_0^2\} \tag{82}$$

$$= 2e^{\mu_1 t} A(\mu_1, \nu_1, h) \cos(\nu_1 t - \phi(\mu_1, \nu_1, h)), \tag{83}$$

where

$$A(\mu_1, \nu_1, h) = \sqrt{\frac{X_1^2 + Y_1^2}{X_0^2 + Y_0^2}} = \frac{|B(\lambda_1, h)|}{|\Delta'(\lambda_1)|}, \tag{84}$$

and

$$\phi(\mu_1, \nu_1, h) = \arctan \frac{Y_1 X_0 - X_1 Y_0}{X_1 X_0 + Y_1 Y_0} = \arctan \frac{\Im(B(\lambda_1, h)/\Delta'(\lambda_1))}{\Re(B(\lambda_1, h)/\Delta'(\lambda_1))}. \tag{85}$$

Note that Dominant Spectral Component $y_0(t)$ is oscillatory with envelope $\pm 2e^{\mu_1 t} A$, phase ϕ, angular frequency ν_1, and with fixed distance π/ν_1 between successive zeros.

THEOREM 8. *Let the roots λ_r of $\Delta(\lambda) = 0$ have leading complex characteristic roots $\lambda_1 = \mu_1 + i\nu_1$ and $\lambda_2 = \mu_1 - i\nu_1$ each with multiplicity 1. Then*

$$\lim_{t\to\infty}\{e^{-\mu_1 t}x(t) - 2A(\mu_1,\nu_1,h)\cos(\nu_1 t - \phi(\mu_1,\nu_1,h))\} = 0. \tag{86}$$

If $\gamma_0 < 0$

$$\lim_{t\to\infty}\{x(t) - 2e^{\mu_1 t}A(\mu_1,\nu_1,h)\cos(\nu_1 t - \phi(\mu_1,\nu_1,h))\} = 0. \tag{87}$$

Proof. We have

$$x(t) = 2e^{\mu_1 t}A(\mu_1,\nu_1,h)\cos(\nu_1 t - \phi(\mu_1,\nu_1,h)) + O(e^{(\gamma_0+\epsilon)t}) \tag{88}$$

The result follows from choosing ϵ such that $\gamma_0 + \epsilon - \mu_1 < 0$ and taking the limit. ∎

Bounds on the distance between adjacent zeros for large semi-cycles (distance greater than the delay a) of the oscillatory solutions of first order RFDE, are given in References [31] and [15]. For $\gamma_0 < 0$ Equation (87) indicates that the zeros of the Solution $x(t)$ asymptotically approach the corresponding zeros of the Dominant Spectral Component $y_0(t)$ and that consequently the distance between adjacent zeros is vanishingly close to π/ν_1. Numerics in Cases 3 and 4 of Section 5 suggests that the distance between consecutive zeros is indeed close to π/ν_1.

For $\gamma_0 > 0$ the sinusoidal functions in the expression for Predominant Spectral Components $y_1(t)$ will have varying exponential growth/decay rates and different angular frequencies and the distance between successive zeros of $x(t)$ is expected to exhibit more complex behavior than for the case $\gamma_0 < 0$.

4. Asymptotic Behavior of Integral of Solution. If the Characteristic Function $\Delta(z)$ has no zeros in the right half plane $\Re z \geq c$ for some real number c it is well known that every Solution $x(t)$ of Equation (23) is exponentially bounded for $t > 0$, i.e. $|x(t)| < Ke^{ct}$ for some constant $K > 0$, and consequently $e^{-ct}x(t)$ is integrable over $[0,\infty)$ with Laplace transform analytic at $z = 0$.

The following Theorem derives the value of the integral of a solution from its Laplace transform.

THEOREM 9. *Let $\Delta(z)$ have no zeros in the right half plane $\Re z \geq c$ for some real number c. Then*

$$\int_0^\infty e^{-ct}\Phi(t)dt = 1/\{c - \int_0^a A(ds)e^{-cs}\}, \tag{89}$$

$$\int_0^\infty e^{-ct}x(t)dt = \frac{B(c,h)}{\Delta(c)} = \frac{h(0) + \int_{-a}^{0} d\alpha\, h(\alpha) \int_{-\alpha}^{a} A(ds)e^{-c(\alpha+s)}}{c - \int_0^a A(ds)e^{-cs}}. \tag{90}$$

Proof. We have $\hat{X}(z) = B(z,h)/\Delta(z)$ and

$$\int_0^\infty e^{-ct}x(t)e^{-zt}dt = \hat{X}(z+c) = B(z+c)/\Delta(z+c). \tag{91}$$

The result for $\int_0^\infty e^{-ct}x(t)dt$ follows from $z = 0$. Similarly

$$\int_0^\infty e^{-ct}\Phi(t)dt = 1/\{c - \int_0^a A(ds)e^{-cs}\}. \tag{92}$$

□

The following corollary follows by putting $c = 0$ in Equations (89) and (90).

COROLLARY 2. *Let $\Delta(z)$ have no zeros in the right half plane $\Re z \geq 0$ so that every solution $x(t)$ is integrable. Then*

$$\int_0^\infty \Phi(t)dt = 1/\big\{-\int_0^a A(ds)\big\}, \tag{93}$$

$$\int_0^\infty x(t)dt = \frac{B(0,h)}{\Delta(0)} = \frac{h(0) + \int_{-a}^{0} d\alpha\, h(\alpha) \int_{-\alpha}^{a} A(ds)}{-\int_0^a A(ds)}. \tag{94}$$

Note that in the case $\Delta(z)$ has a root at 0 that $\int_0^a A(ds) = 0$ and that we need to find a more meaningful expression for the asymptotic behavior for the integral of the Solution $x(t)$. Note that the measure $A(ds)$ has both equal positive and negative decompositions over the interval $[0, a]$, that these cancel each other, so that in some sense the growth is cancelled out in the hereditary mechanics.

The following Theorem relates the asymptotic behavior of the integral $\int_0^T x(t)dt$ as $T \to \infty$ with the behavior of the Laplace transform $\hat{X}(s)$ as $s \to 0$.

THEOREM 10 (Reference [16], page 445). *Let* $\hat{X}(s) = \int_0^\infty e^{-st}x(t)dt$.
The following statement are equivalent:
$\hat{X}(s) \sim Cs^{-\rho}$ *as* $s \to 0$ *for* ρ *non-negative real number.*

$\int_0^T x(t)dt \sim \frac{C}{\Gamma(\rho+1)}T^\rho$ *as* $T \to \infty$,

where the Gamma function $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$.

Note that Corollary 4.2 is based on $\rho = 0$. The next Theorem is based on $\rho = 1$.

THEOREM 11. *Let* $\int_0^a A(ds) = 0$, $1 + \int_0^a A(ds)s \neq 0$,
and $h(0) + \int_{-a}^0 d\alpha\, h(\alpha) \int_{-\alpha}^a A(ds) \neq 0$. *Then*

$$\int_0^T x(t)dt \sim T\big\{h(0) + \int_{-a}^0 d\alpha\, h(\alpha) \int_{-\alpha}^a A(ds)\big\}/\big\{1 + \int_0^a A(ds)s\big\}. \tag{95}$$

Proof.

$$\Delta(z) = z - \int_0^a A(ds)e^{-zs} \tag{96}$$

$$= z - \int_0^a A(ds)\{1 - zs + O((zs)^2\} \tag{97}$$

$$= z\big(1 + \int_0^a A(ds)s\big) + O(z^2). \tag{98}$$

$$\hat{X}(z) \sim z^{-1}\big\{h(0) + \int_{-a}^0 d\alpha\, h(\alpha) \int_{-\alpha}^a A(ds)\big\}/\big\{1 + \int_0^a A(ds)s\big\}. \tag{99}$$

Hence result follows immediately from Theorem 4.3. ∎

5. Numerics for Simple RFDE. As an illustration let us consider the RFDE

$$x'(t) = A_0 x(t) + A_1 x(t-1) \tag{100}$$

with Borel measure $A(ds) = A_0\delta(s) + A_1\delta(s-1)$ and initial data $h(t)$.

The Solution $x(t)$ is given by

$$x(t) = \Phi(t)h(0) + A_1 \int_{-1}^{0} d\alpha\, \Phi(t-1-\alpha)h(\alpha), \tag{101}$$

where the Fundamental Solution $\Phi(t)$ is given by

$$\Phi(t) = \sum_{n=0}^{\infty} \frac{1}{n!} e^{A_0(t-n)} A_1^n (t-n)_+^n \,. \tag{102}$$

The Characteristic Function $\Delta(\lambda)$ is given by

$$\Delta(\lambda) = \lambda - A_0 - A_1 e^{-\lambda}, \tag{103}$$

and the analytical function of the initial data $B(z,h)$ is given by

$$B(z,h) = h(0) + A_1 \int_{-1}^{0} d\alpha\, e^{-z(\alpha+1)} h(\alpha). \tag{104}$$

Note that A_0 does not appear explicitly in $x(t)$, $B(z,h)$, and $\Delta'(\lambda)$. This is true in general for the the size A_0 of the atom at 0 for the measure $A(ds)$.

The Stability-Oscillatory regions R_1, R_2, R_3 and R_4 in the parameter space (A_0, A_1) are shown in Figure 1. The asymptotic characteristics of these regions are:
R_1: Stable/Nonoscillatory,
R_2: Stable/Oscillatory,
R_3: Unstable/Oscillatory,
R_4: Unstable/Nonoscillatory.

The curves C_i defining the regions R_i for i=1,2,3,4 are:

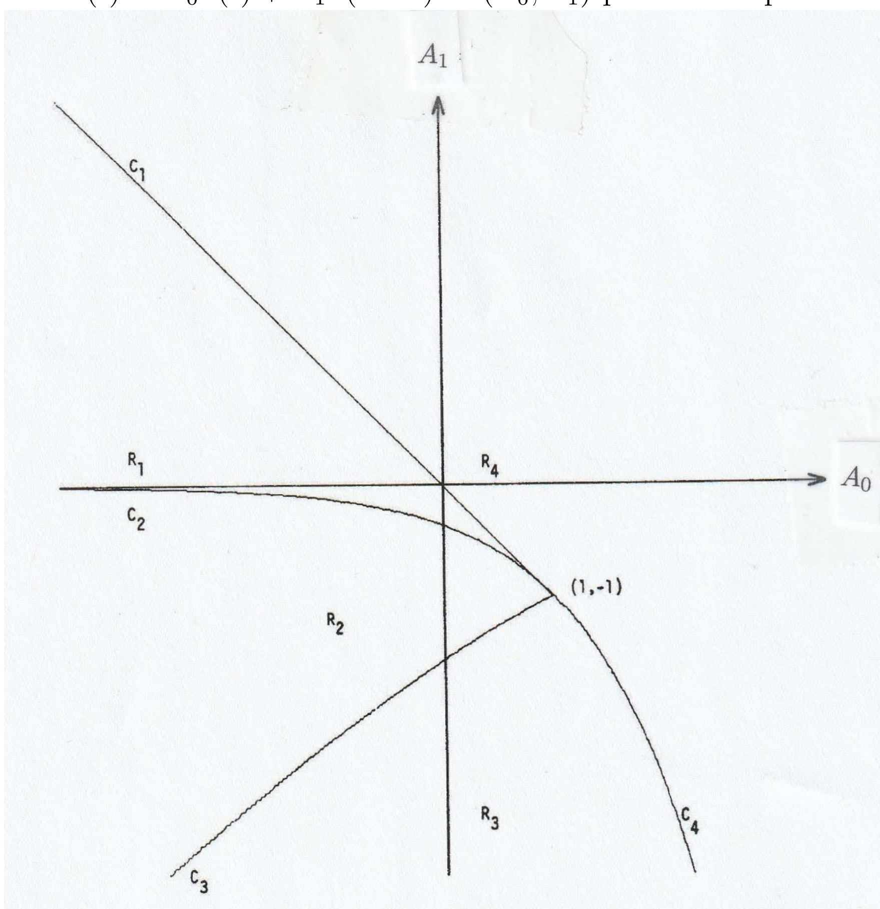
$$C_1 = \{(u,v);\ u + v = 0, u < 1\}, \tag{105}$$

$$C_2 = \{(u,v);\ e^{u-1} + v = 0, u < 1\}, \tag{106}$$

$$C_3 = \{(u,v);\ u = p\cot(p), v = -p\csc(p), 0 < p < \pi\}, \tag{107}$$

$$C_4 = \{(u,v);\ e^{u-1} + v = 0, u \geq 1\}. \tag{108}$$

Figure 1:
Stability-Oscillatory Regions for RFDE
$x'(t) = A_0x(t) + A_1x(t-1)$ in (A_0, A_1) parameter space.



See References [3] (Theorem 13.8 on page 444) and [14] (page 134 and Figures 11 and 16) for the Stability regions, and Reference [32] for the Oscillatory regions.

For the RFDE $x'(t) = A_0x(t) + A_1x(t-1)$ the number of exponential solutions in the Dominant Spectral Component $y_0(t)$ is typically one or two. The number of exponential solutions in the Predominant Spectral Components $y_1(t)$ will depend on the number of characteristic roots in the right half plane. This information is portrayed in the D-Partition diagram of the (A_0, A_1) parameter space in Figure XI.1 on page 306 of Reference [8]. The location of the characteristic roots in the complex plane is described in the context of horizontal strips in the complex plane, such as

$$\Sigma_k = \{(\mu + i\nu);\ \nu \in ((2k-1)\pi, (2k+1)\pi)\}, \tag{109}$$

in Theorems 3.1 and 3.2 on page 312 of Reference [8].

The numerical investigation of the Solution $x(t)$ is conducted with the MATLAB program (see Reference [28]), in particular with the MATLAB dde23 solver (see Reference [29]). Section 12.4 of Reference [22] is a useful guide in setting up the program script.

We use the initial data $h(0) = 1$ and $h(t) = 0$ for $t \in [-1, 0)$ so that we are in effect dealing with the Fundamental Solution $\Phi(t)$. We choose various values of the parameter space (A_0, A_1) to illustrate the full range of asymptotic behavior:

Case (1): $A_0 = 0$, $A_1 = 15$ (Unstable/ Non-Oscillatory).
Leading real root $\lambda_1 = 2.0099$. $\gamma_0 = 1.0889$.
The other characteristic roots in the right half plane are:
1.0889 ± i4.9298, 0.3076 ± i11.0235.
Predominant Spectral Components $y_1(t)$ and Dominant Spectral Component $y_0(t)$ differ. Dominant Spectral Component $y_0(t) = a_1e^{\lambda_1 t}$, where $a_1 = 1/(1+A_1e^{-\lambda_1}) = 0.5399$.
Let $z(t) = e^{-\lambda_1 t}(x(t) - y_0(t)) = e^{-\lambda_1 t}x(t) - a_1$. From Theorem 3.3 we know that $\lim_{t\to\infty} z(t) = 0$. The plot of $z(t)$ is shown in Figure 2.

Case (2): $A_0 = 0$, $A_1 = -0.364$ (Stable/Non-Oscillatory).
Leading real root $\lambda_1 = -0.8614$. $\gamma_0 = -1.1528$.
Predominant Spectral Components $y_1(t)$ same as Dominant Spectral Component $y_0(t)$. Dominant Spectral Component $y_0(t) = a_1e^{\lambda_1 t}$, where $a_1 = 1/(1 + A_1e^{-\lambda_1}) = 7.2148$.
Plots of the Solution $x(t)$ (blue) and the Dominant Spectral Component $y_0(t)$ (red) are shown in Figure 3.

Case (3): $A_0 = 0$, $A_1 = -0.8$ (Stable/Oscillatory).
Leading complex roots $\lambda_1 = \mu_1 + i\nu_1$, $\lambda_2 = \mu_1 - i\nu_1$, $\mu_1 = -0.4730$,

Figure 2:
Plot of $z(t) = e^{-\lambda_1 t}(x(t) - y_0(t))$ for Solution $x(t)$ and Dominant Spectral Component $y_0(t)$ for RDFE $x'(t) = A_0x(t) + A_1x(t-1)$ with $A_0 = 0$, $A_1 = 15$ and initial data $h(0) = 1$ and $h(t) = 0$ for $-a \leq t < 0$.

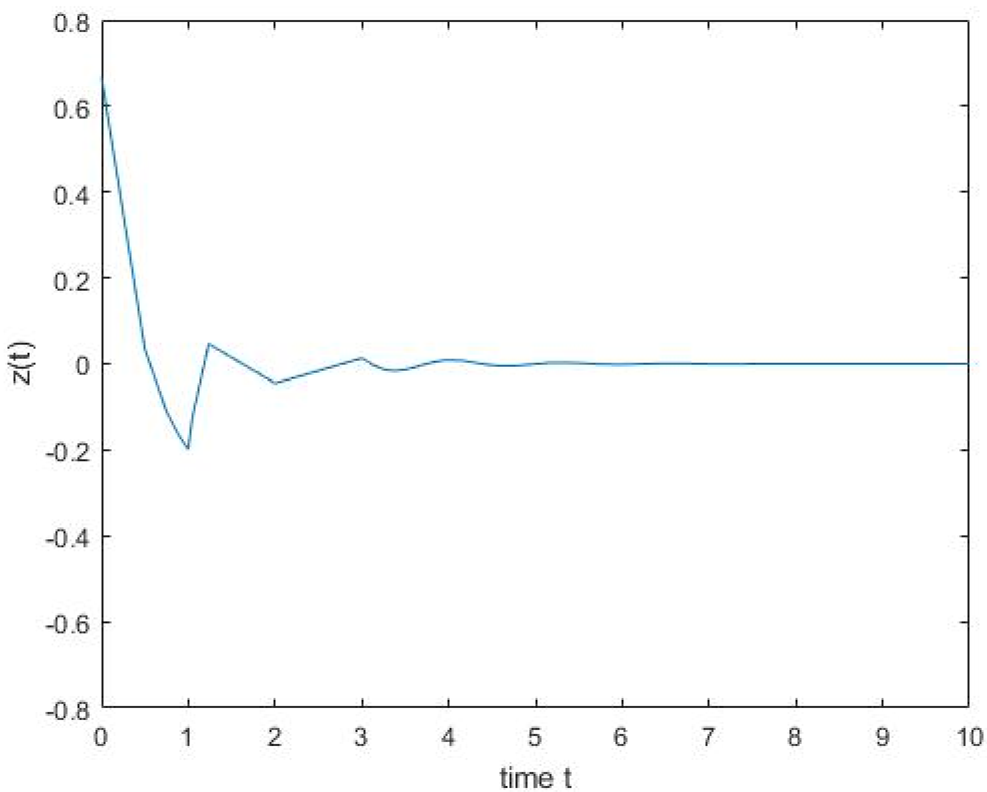


Figure 3:
Plots of Solution $x(t)$ and Dominant Spectral Component $y_0(t)$ for RDFE $x'(t) = A_0x(t) + A_1x(t-1)$ with $A_0 = 0$, $A_1 = -0.364$ and initial data $h(0) = 1$ and $h(t) = 0$ for $-a \leq t < 0$. $x(t)$ and $y_0(t)$ are plotted in blue and red respectively.

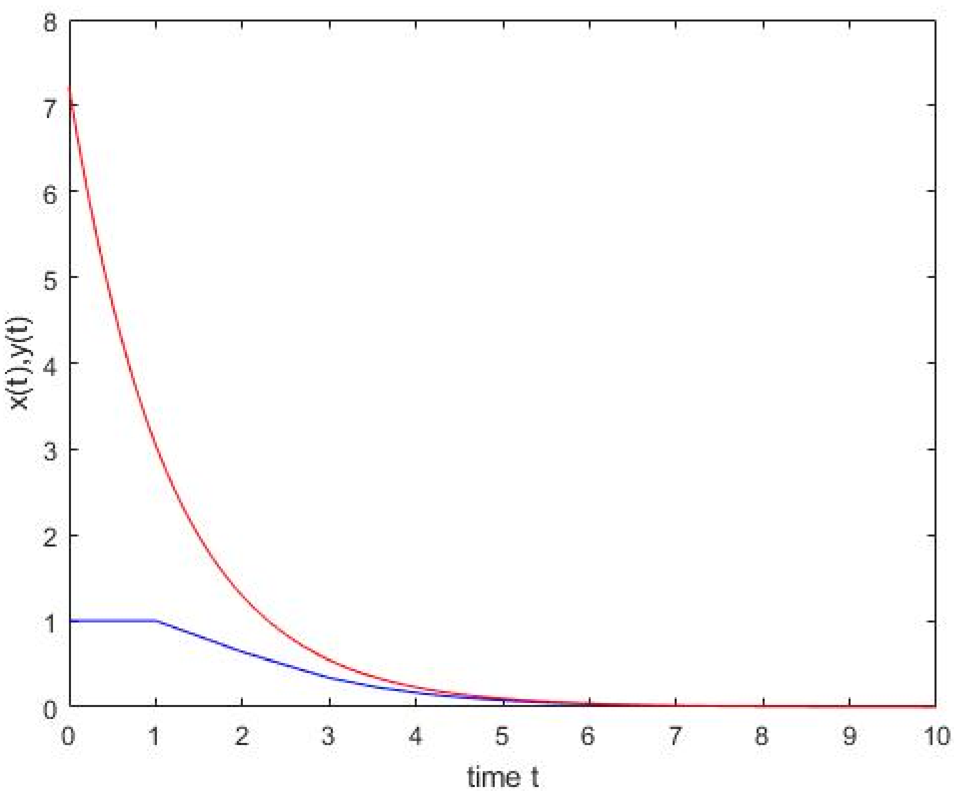
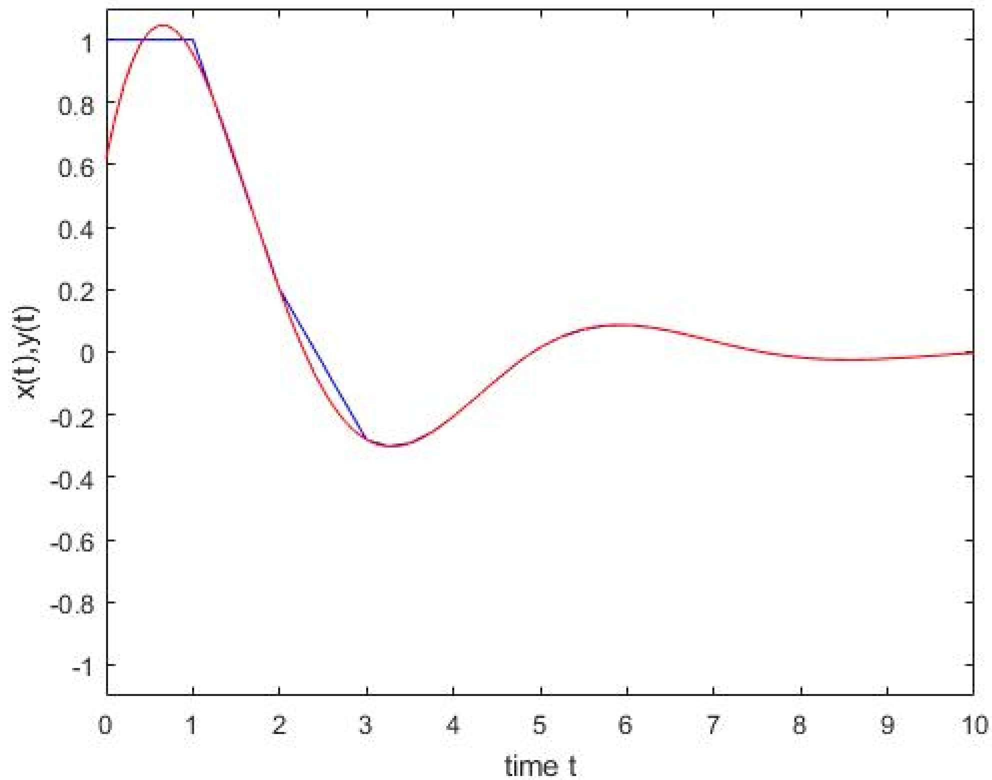


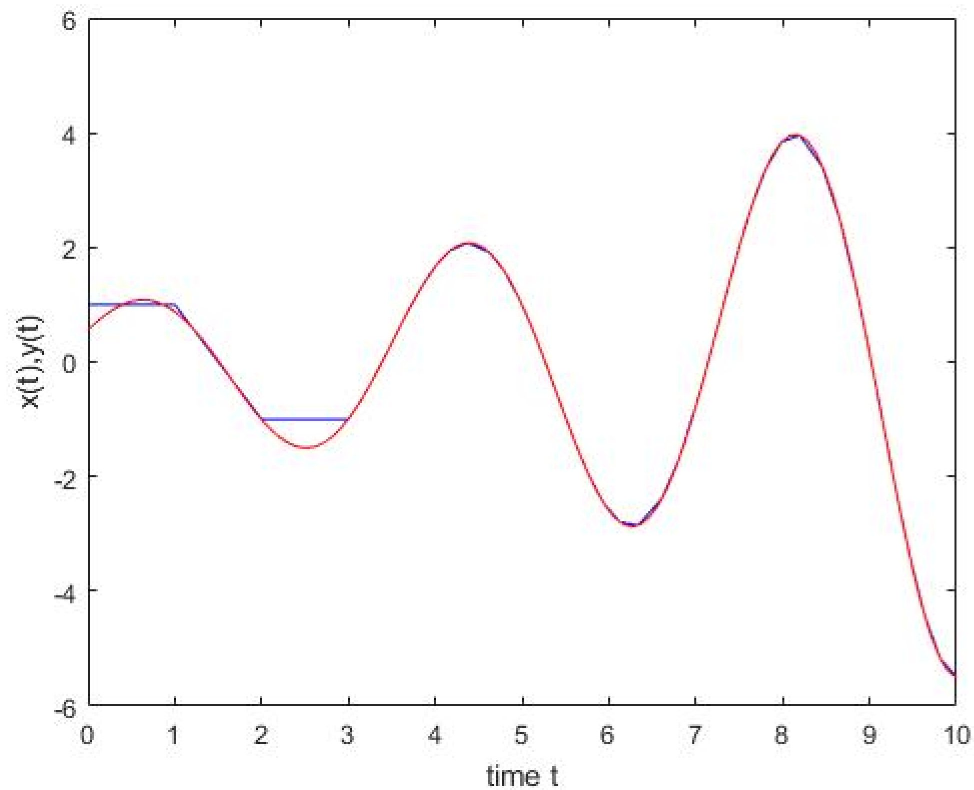
Figure 4:
Plots of Solution $x(t)$ and Dominant Spectral Approximation $y_0(t)$ for RDFE $x'(t) = A_0x(t) + A_1x(t-1)$ with $A_0 = 0$, $A_1 = -0.8$ and initial data $h(0) = 1$ and $h(t) = 0$ for $-a \le t < 0$. $x(t)$ and $y_0(t)$ are plotted in blue and red respectively.



$\nu_1 = 1.1935$. $\gamma_0 = -2.2899$.
Predominant Spectral Components $y_1(t)$ same as Dominant Spectral Component $y_0(t)$. Dominant Spectral Component $y_0(t) = a_1e^{\mu_1 t}\cos(\nu_1 t) + a_2e^{\mu_1 t}\sin(\nu_1 t)$, where $a_1 = 2(1 + A_1e^{-\mu_1}\cos(\nu_1))/(1 + 2A_1e^{-\mu_1}\cos(\nu_1) + A_1^2e^{-2\mu_1})$ and $a_2 = -2A_1e^{-\mu_1}sin(\nu_1)/(1 + 2A_1e^{-\mu_1}\cos(\nu_1) + A_1^2e^{-2\mu_1})$.
Plots of the Solution $x(t)$ (blue) and the Dominant Spectral Component $y_0(t)$ (red) are shown in Figure 4.

Case (4): $A_0 = 0$, $A_1 = -2$ (Unstable/Oscillatory).
Leading complex roots $\lambda_1 = \mu_1 + i\nu_1$, $\lambda_2 = \mu_1 - i\nu_1$, $\mu_1 = 0.1728$, $\nu_1 = 1.6737$. $\gamma_0 = -1.3607$.
Predominant Spectral Components $y_1(t)$ same as Dominant Spectral Component $y_0(t)$. Dominant Spectral Component $y_0(t) = a_1e^{\mu_1 t}\cos(\nu_1 t) + a_2e^{\mu_1 t}\sin(\nu_1 t)$,

Figure 5:
Plots of Solution $x(t)$ and Dominant Spectral Approximation $y_0(t)$ for RDFE $x'(t) = A_0x(t) + A_1x(t-1)$ with $A_0 = 0$, $A_1 = -2$ and initial data $h(0) = 1$ and $h(t) = 0$ for $-a \leq t < 0$. $x(t)$ and $y_0(t)$ are plotted in blue and red respectively



where $a_1 = 2(1 + A_1e^{-\mu_1}\cos(\nu_1))/(1 + 2A_1e^{-\mu_1}\cos(\nu_1) + A_1^2e^{-2\mu_1})$
and $a_2 = -2A_1e^{-\mu_1}\sin(\nu_1)/(1 + 2A_1e^{-\mu_1}\cos(\nu_1) + A_1^2e^{-2\mu_1})$.
Plots of the Solution $x(t)$ (blue) and the Dominant Spectral Component $y_0(t)$ (red) are shown in Figure 5.

Case (5): $A_0 = 0.5$, $A_1 = -0.5$ (Bounded).
This case corresponds to $\int_0^1 A(ds) = 0$ and the RFDE has constant solutions.
Leading real root $\lambda_1 = 0$. $\gamma_0 = -1.2564$.
Predominant Spectral Components $y_1(t)$ same as Dominant Spectral Component $y_0(t)$. Dominant Spectral Component $y_0(t) = 1/1 - A_0 = 2$.
The solution has finite non-zero asymptotic limit. Plots of the Solution $x(t)$ (blue) and the Dominant Spectral Component $y_0(t)$ (red) are shown in Figure 6.

Figure 6:
Plots of Solution $x(t)$ and Dominant Spectral Component $y_0(t)$ for RDFE $x'(t) = A_0x(t) + A_1x(t-1)$ with $A_0 = 0.5$, $A_1 = -0.5$ and initial data $h(0) = 1$ and $h(t) = 0$ for $-a \leq t < 0$. $x(t)$ and $y_0(t)$ are plotted in blue and red respectively.

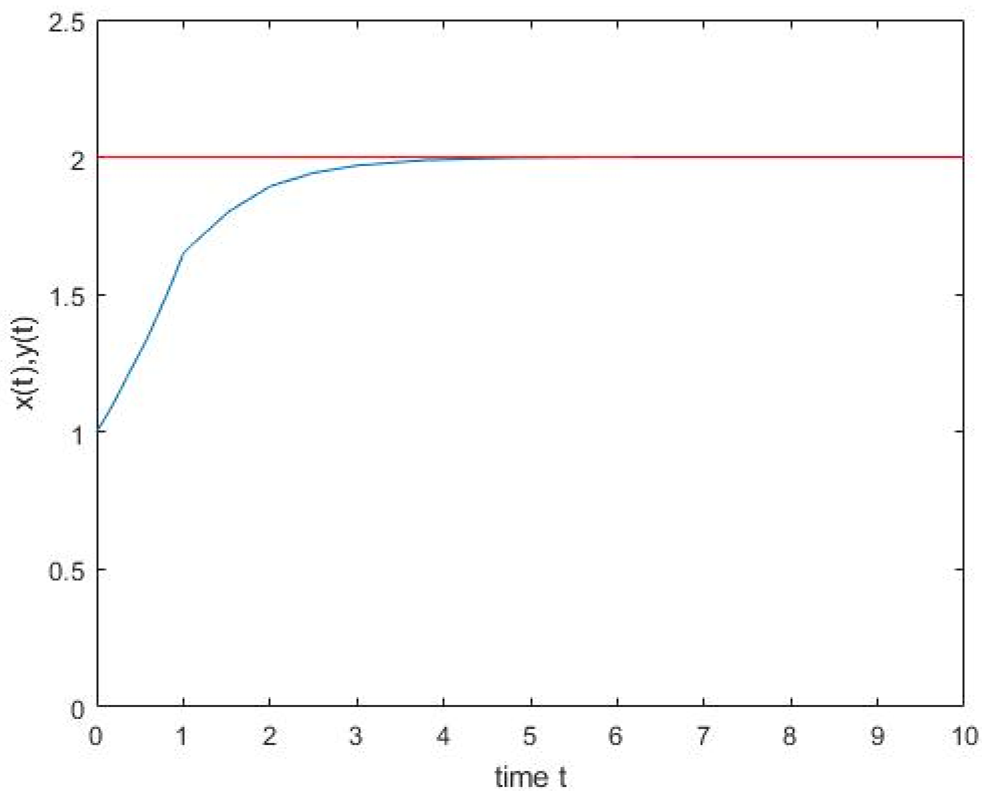
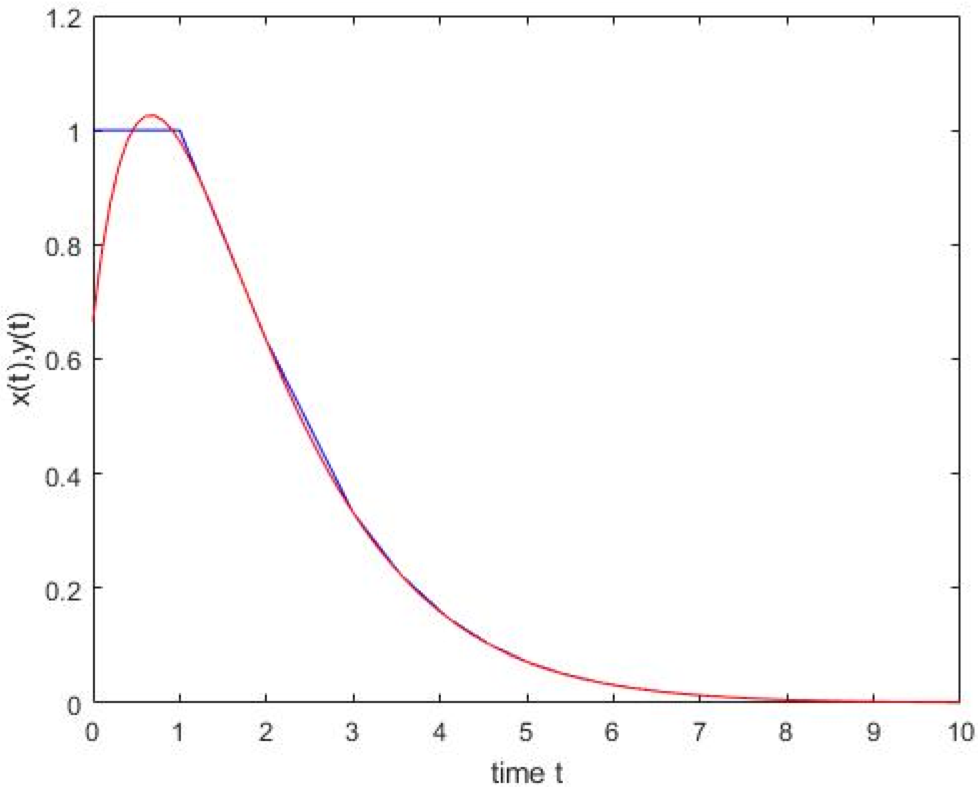


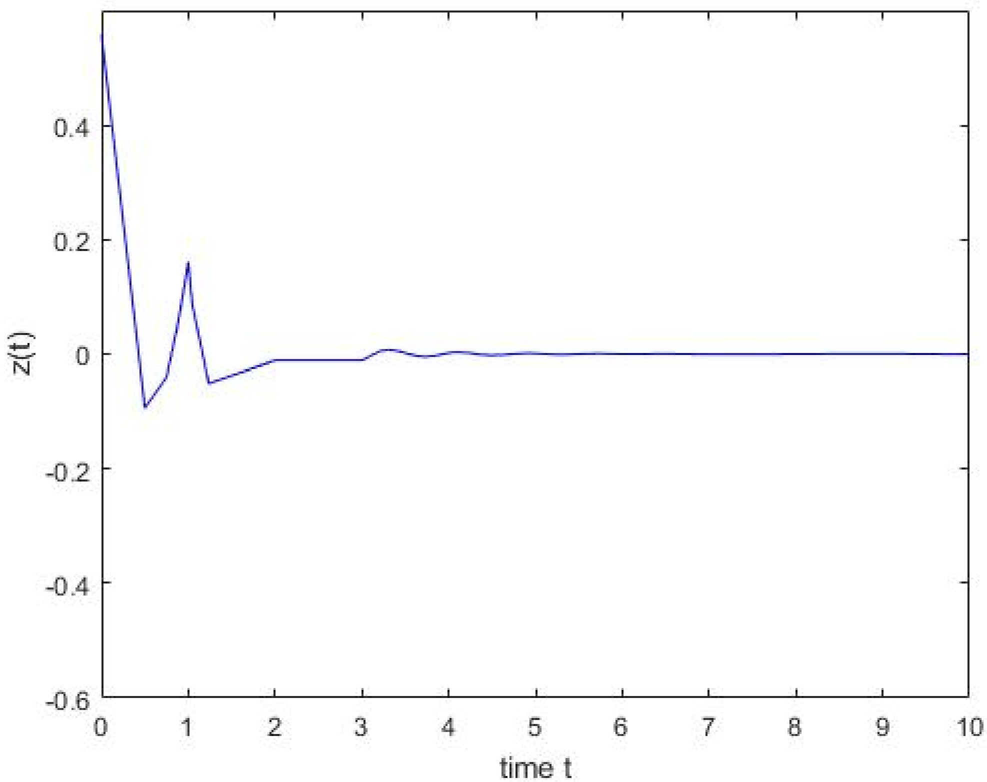
Figure 7:
Plots of Solution $x(t)$ and Dominant Spectral Component $y_0(t)$ for RDFE $x'(t) = A_0x(t) + A_1x(t-1)$ with $A_0 = 0$, $A_1 = -e^{-1}$ and initial data $h(0) = 1$ and $h(t) = 0$ for $-a \leq t < 0$. Characteristic root $\lambda_1 = -1$ with multiplicity 2. $x(t)$ and $y_0(t)$ are plotted in blue and red respectively.



Case (6): $A_0 = 0$, $A_1 = -e^{-1} = -0.3679$ (Stable/Critical Nonoscillatory).
Leading real root $\lambda_1 = -1$ with multiplicity 2. $\gamma_0 = -3.0888$.
Predominant Spectral Components $y_1(t)$ is same as Dominant Spectral Component $y_0(t)$. Dominant Spectral Component $y_0(t) = 2e^{-t}(t + 1/3)$.
Plots of the Solution $x(t)$ (blue) and the Dominant Spectral Component $y_0(t)$ (red) are shown in Figure 7.

Case (7): $A_0 = 0$, $A_1 = -15$ (Unstable/Oscillatory).
Leading complex roots $\lambda_1 = \mu_1 + i\nu_1$, $\lambda_2 = \mu_1 - i\nu_1$, $\mu_1 = 1.6835$, $\nu_1 = 2.2197$. $\gamma_0 = 0.6338$.
The other characteristic roots in the right half plane are:
$0.6338 \pm i7.9337$, $0.0589 \pm i14.1413$.

Figure 8:
Plot of $z(t) = e^{-\mu_1 t}(x(t) - y_0(t))$ for Solution $x(t)$ and Dominant Spectral Component $y_0(t)$ for RDFE $x'(t) = A_0 x(t) + A_1 x(t-1)$
with $A_0 = 0$, $A_1 = -15$ and initial data $h(0) = 1$ and $h(t) = 0$ for $-a \le t < 0$.



Predominant Spectral Components $y_1(t)$ and Dominant Spectral Component $y_0(t)$ differ. Dominant Spectral Component $y_0(t) = a_1 e^{\mu_1 t}\cos(\nu_1 t) + a_2 e^{\mu_1 t}\sin(\nu_1 t)$,
where $a_1 = 2(1 + A_1 e^{-\mu_1}\cos(\nu_1))/(1 + 2A_1 e^{-\mu_1}\cos(\nu_1) + A_1^2 e^{-2\mu_1})$
and $a_2 = -2A_1 e^{-\mu_1}\sin(\nu_1)/(1 + 2A_1 e^{-\mu_1}\cos(\nu_1) + A_1^2 e^{-2\mu_1})$.
Let $z(t) = e^{-\mu_1 t}(x(t) - y_0(t))$. From Theorem 3.6 we know that $\lim_{t\to\infty} z(t) = 0$. The plot of $z(t)$ is shown in Figure 8.

For the oscillatory Cases (3,4,7), Theorem 3.2 of Reference [8] states that $\nu_1 \in (0, \pi)$ so that $\pi/\nu_1 > 1 = a$ and the semicycles of the Dominant Spectral Component $y_0(t)$ are large. For the remaining spectral components with $k \ge 2$ $\nu_k \in ((2k-2)\pi, (2k-1)\pi)$ so that $\pi/\nu_k < 1/(2k-2) < 1)$ and the semicycles are small.

Figures 2-8 corresponding to Cases 1-7 demonstrate that the transitional period to asymptotic behavior is fairly rapid, after a few delays. This follows directly from the exponential convergence, such as $x(t) - y_0(t) = O(e^{(\gamma_0+\epsilon)t})$ for $\gamma_0 \le 0$. For the bound in the difference (relative or absolute) in asymptotic behavior given by the exponential function Ke^{-kt} define the transitional period T_p by $Ke^{-kT_p} = 10^{-2}$ so that $T_p \approx 4.6/k$ ignoring $\ln(K)/k$. In Case(1) $k = \lambda_1 - \gamma_0 = 0.921$ and $T_p = 5.0$, which is consistent with Figure 2. In Case(6), $k = -\gamma_0 = 3.088$ and $T_p = 1.5$, which is consistent with Figure 7.

For the RFDE $x'(t) = A_1x(t-1)$ with initial condition $h(t) = 1$ for $t \in [0,1]$ and $-5\pi/2 \le A_1 \le 3\pi/2$ (corresponding to Cases 2-4,6), Reference [13] provides an analysis to obtain a bound on the error $g(t) = x(t) - y_0(t)$ in the form $|g(t)| \le K\exp(-kt)$ for specified values of $K > 0$ in addition to $k > 0$. This produces a more refined estimate of the transitional period to asymptotic convergence.

Note that the solution of $x'(t) = A_1x(t-1)$ with initial data $h(t) = 1$ for $t \in [0,1]$ is $x(t) = \Phi(t+1)$, since this initial data corresponds to $\Phi(t)$ for $t \in [0,1]$. Let $g_0(t)$ and $g_1(t)$ be the difference between the solution and the asymptotic exponential approximation respectively for the initial datum A: $h(t) = 0$ for $t \in [-1,0)$, $h(0) = 1$ and B: $h(t) = 1$ for $h \in [-1,0]$. Then

$$g_1(t) = \Phi(t+1) - \exp(\lambda_1 t)\exp(\lambda_1)/(1 + A_1\exp(-\lambda_1) = g_0(t+1) \tag{110}$$

so that the transitional period to asymptotic stability for the RDFEs studied in Reference [13] for initial data B is one delay time less than that shown in Figures 3-5,7 due to the different choice in initial data. The point is that the transitional period to asymptotic convergence depends on the initial data. The closer the initial data is to the asymptotic exponential solution, the quicker is the transitional period.

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