ON POSITIVITY OF GREEN’S FUNCTIONS OF TWO-POINT IMPULSIVE PROBLEMS

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Abstract. We consider the following second order impulsive differential equation with delays

\[
\begin{align*}
(Lx)(t) & \equiv x''(t) + \sum_{j=1}^{p} a_j(t)x'(t - \tau_j(t)) + \sum_{j=1}^{p} b_j(t)x(t - \theta_j(t)) = f(t), \quad t \in [0, \omega], \\
x(t_k) & = \gamma_k x(t_k - 0), \quad x'(t_k) = \delta_k x'(t_k - 0), \quad k = 1, 2, \ldots, r.
\end{align*}
\]

In this paper we develop the approach of [5] and obtain explicit conditions of nonpositivity of Green’s functions for two-point boundary value problems.

Key Words. Second order impulsive differential equations, boundary value problems, sign-constancy of Green’s functions

AMS(MOS) subject classification. 34K06, 34K10, 34K45

1. Introduction. Impulsive differential equations has attracted an attention of a number of recognized mathematicians and has applications in many spheres of science from physics, biology, medicine to economical studies. The following well-known books can be noted in this context [11, 13, 14, 15]. In the book [3] the concept of the general theory of functional differential equations was presented. On the basis of this concept nonoscillation for the first order functional differential equations was considered in [4], where positivity of the Cauchy and Green’s functions of the periodic problem was studied. A concept of nonoscillation for the first order differential equations is also considered in the book [1]. The positivity of

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Green’s function of one- and two-point boundary value problems was studied in [2, 5, 6, 7, 8, 9, 10, 12].

This paper develops the approach of [5] and is aimed to obtain explicit conditions of nonpositivity of Green’s functions for two-point boundary value problems.

Let us consider the following impulsive equations:

\[(Lx)(t) \equiv x''(t) + \sum_{j=1}^{p} a_j(t)x'(t - \tau_j(t)) + \sum_{j=1}^{p} b_j(t)x(t - \theta_j(t)) = f(t),\]

\[t \in [0, \omega],\]

\[x(t_k) = \gamma_k x(t_k - 0), \quad x'(t_k) = \delta_k x'(t_k - 0), \quad k = 1, 2, ..., p,\]

\[0 = t_0 < t_1 < t_2 < ... < t_r < t_{r+1} = \omega,\]

\[x(\zeta) = 0, \quad \zeta < 0,\]

where \(f, a_j, b_j: [0, \omega] \to \mathbb{R}\) are summable functions and \(\tau_j, \theta_j: [0, \omega] \to [0, +\infty)\) are measurable functions for \(j = 1, 2, ..., p, p\) and \(r\) are natural numbers, \(\gamma_k\) and \(\delta_k\) are real positive numbers.

Let \(D(t_1, t_2, ..., t_r)\) be a space of functions \(x: [0, \omega] \to \mathbb{R}\) such that their derivative \(x'(t)\) is absolutely continuous on every interval \(t \in [t_i, t_{i+1})\), \(i = 0, 1, ..., r\), \(x''(t) \in L_\infty\) there exist the finite limits \(x(t_i - 0) = \lim_{t \to t_i^-} x(t)\) and \(x'(t_i - 0) = \lim_{t \to t_i^-} x'(t)\) and condition (2) is satisfied at points \(t_i\) \((i = 0, 1, ..., r)\). Solution \(x\) is a function \(x \in D(t_1, t_2, ..., t_r)\) satisfying (1)-(3).

2. Preliminaries. For equation (1)-(3) we consider the following variants of boundary conditions:

\[(4) \quad x(0) = 0, \quad x(\omega) = 0,\]

\[(5) \quad x'(0) = 0, \quad x(\omega) = 0,\]

\[(6) \quad x(0) = 0, \quad x'(\omega) = 0,\]

\[(7) \quad x'(0) = 0, \quad x'(\omega) = 0.\]

General solution of the equation (1)-(3) can be represented in the form [4]:

\[(8) \quad x(t) = \nu_1(t)x(0) + C(t, 0)x'(0) + \int_0^t C(t, s)f(s)ds,\]

where
\( \nu_1(t) \) is a solution of the homogeneous equation

\[
(Lx)(t) \equiv x''(t) + \sum_{j=1}^{p} a_j(t)x'(t - \tau_j(t)) + \sum_{j=1}^{p} b_j(t)x(t - \theta_j(t)) = 0, \quad t \in [0, \omega],
\]

(9)

\( x(t_k) = \gamma_k x(t_k - 0), \quad x'(t_k) = \delta_k x'(t_k - 0), \quad k = 1, 2, ..., r, \)

\[
0 = t_0 < t_1 < t_2 < ... < t_r < t_{r+1} = \omega,
\]

(10)

\( x(\zeta) = 0, \quad \zeta < 0, \)

with the boundary conditions \( x(0) = 1, \; x'(0) = 0. \)

\( C(t, s) \) is a Cauchy function of the equation (9)-(11).

It means that \( C(t, s) \) is the solution of the equation

\[
(L_s x)(t) \equiv x''(t) + \sum_{j=1}^{p} a_j(t)x'(t - \tau_j(t)) + \sum_{j=1}^{p} b_j(t)x(t - \theta_j(t)) = 0, \quad t \in [s, \omega],
\]

(12)

\( x(t_k) = \gamma_k x(t_k - 0), \quad x'(t_k) = \delta_k x'(t_k - 0), \quad k = 1, 2, ..., r, \)

\[
0 = t_0 < t_1 < t_2 < ... < t_r < t_{r+1} = \omega,
\]

(13)

\( x(\zeta) = 0, \quad \zeta < s, \)

satisfying the conditions \( C(s, s) = 0, \; C'(s, s) = 1, \; C(t, s) = 0 \) for \( t < s. \)

If the boundary value problem (1)-(3), (2.i), \( i = 1, 4 \) is uniquely solvable, then its solution can be represented as

\[
x(t) = \int_0^{\omega} G_i(t, s)f(s)ds, \quad i = 1, 4,
\]

(15)

where \( G_i(t, s) \) is Green’s function of the problem (1)-(3), (2.i) respectively [5].
Using general representation of the solution (8), the following formulas for Green’s functions can be obtained:

\[(16) \quad G_1(t, s) = C(t, s) - C(t, 0) \frac{C(\omega, s)}{C(\omega, 0)}, \]

\[(17) \quad G_2(t, s) = C(t, s) - C(\omega, s) \frac{\nu_1(t)}{\nu_1(\omega)}, \]

\[(18) \quad G_3(t, s) = C(t, s) - C(t, 0) \frac{C'_1(\omega, s)}{C'_1(\omega, 0)}, \]

\[(19) \quad G_4(t, s) = C(t, s) - C'_1(\omega, s) \frac{\nu_1(t)}{\nu_1(\omega)}. \]

Below the following definition will be used.

**Definition 1.** We call \([0, \omega]\) a semi-nonoscillation interval of (9)-(11), if every nontrivial solution having zero of derivative does not have zero on this interval.

In the paper [5] the following assertions about test functions have been proven.

**Lemma 1.** Assume that \(a_j(t) \geq 0, b_j(t) \leq 0\) for \(j = 1, \ldots, p, 0 < \gamma_k \leq 1, 0 < \delta_k \leq 1\) for \(k = 1, \ldots, r\), and there exists a function \(v \in D\) and \(\epsilon > 0\) such that

\[(20) \quad (Lv)(t) \geq \epsilon > 0, \quad v(t) > 0, \quad v'(t) < 0, \quad v''(t) > 0, \quad t \in (0, \omega),\]

where the differential operator \(L\) is defined by (1). And let \([0, \omega]\) be a semi-nonoscillation interval of (9)-(11). Then Green’s functions \(G_1(t, s), G_2(t, s), G_3(t, s)\) satisfy the inequalities \(G_1(t, s) \leq 0, G_2(t, s) \leq 0, G_3(t, s) \leq 0, (t, s) \in [0, \omega] \times [0, \omega]\). If, in addition, \(\sum_{j=1}^{p} b_j(t) \chi_{[0, \omega]}(t - \theta_j(t)) \neq 0, t \in [0, \omega]\), then \(G_4(t, s) \leq 0, (t, s) \in [0, \omega] \times [0, \omega]\).

**Lemma 2.** Assume that \(a_j(t) \geq 0, b_j(t) \geq 0\) for \(j = 1, \ldots, p, 1 \leq \gamma_k, 1 \leq \delta_k\) for \(k = 1, \ldots, r\), and there exists a function \(v \in D\) and \(\epsilon > 0\) such that

\[(21) \quad (Lv)(t) \leq -\epsilon < 0, \quad v(t) > 0, \quad v'(t) > 0, \quad v''(t) < 0, \quad t \in (0, \omega),\]

where the differential operator \(L\) is defined by (1). And let \([0, \omega]\) be a semi-nonoscillation interval of (9)-(11). Then Green’s function \(G_1(t, s)\) satisfies the inequality \(G_1(t, s) \leq 0, (t, s) \in [0, \omega] \times [0, \omega]\).
A question, connected with a particular form of the test functions, satisfying the conditions of Lemma 1 and Lemma 2, will be considered in the next section.

3. Construction of test functions. Let us now find an example of a function \( v(t) \) satisfying the condition of Lemma 1. Let us start with \( v(t) = e^{-\alpha t} \) in the interval \( t \in [0, t_1) \). The function \( v(t) \) in the rest of the intervals will be of the form

\[
(22) \quad v(t) = c_i e^{-\alpha d_i t}, \quad t \in [t_i, t_{i+1}),
\]

\( q, c_i, d_i \in \mathbb{R}, c_i \neq 0, i = 1, 4 \) and the conditions (2) are fulfilled.

After some calculations, we get that \( v(t) \) is of the form

\[
(23) \quad \begin{cases} 
  v(t) = e^{-\alpha t}, & t \in [0, t_1), \\
  v(t) = \prod_{j=1}^{i} \gamma_j e^{-\alpha \prod_{k=1}^{i} \delta_j (t-t_k)} \prod_{j=1}^{i-1} e^{-\alpha \prod_{k=1}^{i} \delta_j (t_{j+1}-t_j)} e^{-\alpha t}, & t \in [t_i, t_{i+1}).
\end{cases}
\]

It should be mentioned that in the paper [5] the form of the function \( v(t) \) (see, equation (3.12) from [5]) covers only the case when \( \gamma_i = \delta_i, \ i = 1, ..., r \). But our formula (23) is fair for any \( \gamma_i > 0 \) and \( \delta_i > 0, \ i = 1, ..., r \).

For the next theorems, we use the following notation:

\[
(24) \quad E = \min_{i=1,2,...,r} \prod_{j=1}^{i} \frac{\delta_j}{\gamma_j},
\]

\[
(25) \quad \Theta = \max_{t \in [0,\omega]} \max_{j=1,2,...,p} \theta_j(t),
\]

\[
(26) \quad T = \max_{t \in [0,\omega]} \max_{j=1,2,...,p} \tau_j(t),
\]

\[
(27) \quad \Omega = \max\{T, \Theta\}.
\]

In the theorems below we assume that the delays are small enough to stay within the current interval, i.e. \( t - \tau_j \in [t_i, t_{i+1}) \) and \( t - \theta_j \in [t_i, t_{i+1}) \).
Theorem 1. Let \([0, \omega]\) be a semi-nonoscillation interval of (9)-(11). If \(a_j \geq 0, b_j \leq 0, 0 < \gamma_i < \delta_i \leq 1, t_i \leq t < t_{i+1}, t_i \leq t - \tau_j(t) < t_{i+1}, \) for \(t \in (t_i, t_{i+1}), j = 1, \ldots, p, i = 0, \ldots, r, \) and

\[
(28) \quad \alpha^2 E^2 e^{-\alpha \Omega} - \alpha E \text{esssup}_{t \in [0, \omega]} \sum_{j=1}^p |a_j(t)| > \text{esssup}_{t \in [0, \omega]} \sum_{j=1}^p |b_j(t)|,
\]

where \(\alpha\) satisfies the equation:

\[
(29) \quad \alpha(2 - \alpha E \Omega)e^{-\alpha \Omega} = \frac{1}{E} \text{esssup}_{t \in [0, \omega]} \sum_{j=1}^p |a_j(t)|,
\]

then Green's functions \(G_1(t, s), G_2(t, s), G_3(t, s)\) are nonpositive. If, in addition, \(\sum_{j=1}^p b_j(t) \chi_{[0, \omega]}(t - \theta_j(t)) \neq 0\), then Green's function \(G_4(t, s)\) is also nonpositive.

Proof. Let us substitute this \(v(t)\), defined by (23), into the condition of Lemma 1. We obtain:

\[
(30) \quad \alpha^2 \left( \prod_{j=1}^i \frac{\delta_j}{\gamma_j} \right)^2 - \alpha \prod_{j=1}^i \frac{\delta_j}{\gamma_j} \text{esssup}_{t \in [0, \omega]} \sum_{j=1}^p |a_j(t)| e^{\alpha \prod_{j=1}^i \frac{\delta_j}{\gamma_j} \tau_j(t)} - \text{esssup}_{t \in [0, \omega]} \sum_{j=1}^p |b_j(t)| e^{\alpha \prod_{j=1}^i \frac{\delta_j}{\gamma_j} \theta_j(t)} > 0.
\]

Thus,

\[
(31) \quad \alpha^2 E^2 e^{-\alpha \Omega} - \alpha E \text{esssup}_{t \in [0, \omega]} \sum_{j=1}^p |a_j(t)| > \text{esssup}_{t \in [0, \omega]} \sum_{j=1}^p |b_j(t)|,
\]

where \(E\) and \(\Omega\) are defined by (24) and (27), correspondingly. Denoting

\[
(32) \quad F(\alpha) = \alpha^2 E^2 e^{-\alpha \Omega} - \alpha E \text{esssup}_{t \in [0, \omega]} \sum_{j=1}^p |a_j(t)|,
\]

we can find its maximum using the derivative:

\[
(33) \quad F'(\alpha) = \alpha E^2 (2 - \alpha E \Omega)e^{-\alpha \Omega} - E \text{esssup}_{t \in [0, \omega]} \sum_{j=1}^p |a_j(t)|.
\]

The solution of the equation

\[
(34) \quad \alpha(2 - \alpha E \Omega)e^{-\alpha \Omega} = \frac{1}{E} \text{esssup}_{t \in [0, \omega]} \sum_{j=1}^p |a_j(t)|
\]
will give us a point of maximum.

Using this $\alpha$ in Lemma 1, we obtain a condition of nonpositivity of Green’s functions $G_1(t, s), G_2(t, s), G_3(t, s)$. If, in addition, $\sum_{j=1}^{p} b_j(t) \chi_{[0, \omega]}(t-\theta_j(t)) \neq 0$, $t \in [0, \omega]$, then, according to Lemma 1, $G_4(t, s) \leq 0$.

\[ \square \]

**Example 1.** Let $r = 1, \gamma_1 = 0.5, \delta_1 = 0.8, t_1 = 0.5, \omega = 1, p = 1$ and delays satisfy the conditions $t_i \leq t - \tau_1(t) < t_{i+1}, t_i \leq t - \theta_1(t) < t_{i+1}$, for $t \in (t_i, t_{i+1}), i = 0, ..., r$. Let also $a_1(t) = 0.3t$.

Then $\Omega$ can not exceed 0.5 and $E = 1.6$. Substituting these values into the equation (29) and solving it, we get $\alpha = 1.9311$.

Then the condition (28) can be written in the form:

\begin{equation}
\esssup_{t \in [0, \omega]} \sum_{j=1}^{p} |b_j(t)| < 1.1097.
\end{equation}

If $a_j(t)$ is close to zero for $t \in [0, \omega], j = 1, ..., p$, then the following Corollary is fulfilled.

**Corollary 1.** Let $[0, \omega]$ be a semi-nonoscillation interval of (9)-(11). If $a_j \approx 0, b_j \leq 0, 0 < \gamma_i < \delta_i \leq 1, t_i \leq t - \tau_j(t) < t_{i+1}, t_i \leq t - \theta_j(t) < t_{i+1}$, for $t \in (t_i, t_{i+1}), j = 1, ..., p, i = 0, ..., r$, and

\begin{equation}
4 \frac{e^{-2}}{i^2} > \esssup_{t \in [0, \omega]} \sum_{j=1}^{p} |b_j(t)|,
\end{equation}

then Green’s functions $G_1(t, s), G_2(t, s), G_3(t, s)$ are nonpositive. If, in addition, $\sum_{j=1}^{p} b_j(t) \chi_{[0, \omega]}(t-\theta_j(t)) \neq 0$, then Green’s function $G_4(t, s)$ is also nonpositive.

**Example 2.** Let $r = 1, \gamma_1 = 0.5, \delta_1 = 0.8, t_1 = 0.5, \omega = 1, p = 1$ and delays satisfy the conditions $t_i \leq t - \tau_1(t) < t_{i+1}, t_i \leq t - \theta_1(t) < t_{i+1}$, for $t \in (t_i, t_{i+1}), i = 0, ..., r$. Let also $a_1(t) = 0$.

Then $\Omega$ can not exceed 0.5 and $E = 1.6$. Substituting these values into the equation (29) and solving it, we get $\alpha = 2.5$.

Then the condition (28) can be written in the form:

\begin{equation}
\esssup_{t \in [0, \omega]} \sum_{j=1}^{p} |b_j(t)| < 2.1653.
\end{equation}
Theorem 2. Let \([0, \omega]\) be a semi-nonoscillation interval of (9)-(11). If \(a_j \geq 0, b_j \leq 0, 0 < \delta_i \leq \gamma_i \leq 1, t_i \leq t - \tau_j(t) \leq t_{i+1}, t_i \leq t - \theta_j(t) \leq t_{i+1},\) for \(t \in (t_i, t_{i+1})\), \(j = 1, \ldots, p, i = 0, \ldots, r,\) and

(38) \[\alpha^2 E^2 e^{-\alpha \Omega} - \alpha E \mathop{\text{esssup}}_{t \in [0, \omega]} \sum_{j=1}^{p} |a_j(t)| > \mathop{\text{esssup}}_{t \in [0, \omega]} \sum_{j=1}^{p} |b_j(t)|,\]

where \(\alpha\) satisfies the equation:

(39) \[\alpha(2 - \alpha \Omega)e^{-\alpha \Omega} = \frac{1}{E} \mathop{\text{esssup}}_{t \in [0, \omega]} \sum_{j=1}^{p} |a_j(t)|,\]

then Green’s functions \(G_1(t, s), G_2(t, s), G_3(t, s)\) are nonpositive. If, in addition, \(\sum_{j=1}^{p} b_j(t) \chi_{[0, \omega]}(t - \theta_j(t)) \neq 0,\) then Green’s function \(G_4(t, s)\) is also nonpositive.

Proof. Let us substitute this \(v(t)\), defined by (23), into the condition of Lemma 1. We obtain:

(40) \[\alpha^2 \left(\prod_{j=1}^{i} \frac{\delta_j}{\gamma_j}\right)^2 - \alpha \prod_{j=1}^{i} \frac{\delta_j}{\gamma_j} \mathop{\text{esssup}}_{t \in [0, \omega]} \sum_{j=1}^{p} |a_j(t)| e^{\alpha \tau_j(t)} - \mathop{\text{esssup}}_{t \in [0, \omega]} \sum_{j=1}^{p} |b_j(t)| e^{\alpha \theta_j(t)} > 0.\]

Thus,

(41) \[\alpha^2 E^2 e^{-\alpha \Omega} - \alpha E \mathop{\text{esssup}}_{t \in [0, \omega]} \sum_{j=1}^{p} |a_j(t)| > \mathop{\text{esssup}}_{t \in [0, \omega]} \sum_{j=1}^{p} |b_j(t)|,\]

where \(E\) and \(\Omega\) are defined by (24) and (27), correspondingly. Denoting

(42) \[F(\alpha) = \alpha^2 E^2 e^{-\alpha \Omega} - \alpha E \mathop{\text{esssup}}_{t \in [0, \omega]} \sum_{j=1}^{p} |a_j(t)|,\]

we can find its maximum using the derivative:

(43) \[F'(\alpha) = \alpha E(2 - \alpha \Omega)e^{-\alpha \Omega} - \alpha E \mathop{\text{esssup}}_{t \in [0, \omega]} \sum_{j=1}^{p} |a_j(t)|.\]

The solution of the equation

(44) \[\alpha(2 - \alpha \Omega)e^{-\alpha \Omega} = \frac{1}{E} \mathop{\text{esssup}}_{t \in [0, \omega]} \sum_{j=1}^{p} |a_j(t)|\]
will give us a point of maximum.

Using this $\alpha$ in Lemma 1, we obtain a condition of nonpositivity of Green's functions $G_1(t, s), G_2(t, s), G_3(t, s)$. If, in addition, $\sum_{j=1}^{p} b_j(t) \chi_{[0, \omega]}(t - \theta_j(t)) \neq 0$, $t \in [0, \omega]$, then, according to Lemma 1, $G_4(t, s) \leq 0$.

\[ \square \]

**Example 3.** Let $r = 1$, $\gamma_1 = 0.8$, $\delta_1 = 0.5$, $t_1 = 0.5$, $\omega = 1$, $p = 1$ and delays satisfy the conditions $t_i \leq t - \tau_i(t) < t_{i+1}$, $t_i \leq t - \theta_i(t) < t_{i+1}$, for $t \in (t_i, t_{i+1})$, $i = 0, ..., r$. Let also $a_1(t) = 0.3t$.

Then $\Omega$ can not exceed 0.5 and $E = 1.6$. Substituting these values into the equation (2) and solving it, we get $\alpha = 2.6389$.

Then the condition (38) can be written in the form:

\[
\underset{t \in [0, \omega]}{\text{esssup}} \sum_{j=1}^{p} |b_j(t)| < 0.2322.
\]

If $a_j(t)$ is close to zero for $t \in [0, \omega]$, $j = 1, ..., p$, then the following Corollary is fulfilled.

**Corollary 2.** Let $[0, \omega]$ be a semi-nonoscillation interval of (9)-(11). If $a_j \approx 0$, $b_j \leq 0$, $0 < \delta_i \leq \gamma_i \leq 1$, $t_i \leq t - \tau_j(t) \leq t_{i+1}$, $t_i \leq t - \theta_j(t) \leq t_{i+1}$, for $t \in (t_i, t_{i+1})$, $j = 1, ..., p$, $i = 0, ..., r$, and

\[
\frac{4E^2}{\Omega^2} e^{-2} > \underset{t \in [0, \omega]}{\text{esssup}} \sum_{j=1}^{p} |b_j(t)|,
\]

then Green's functions $G_1(t, s), G_2(t, s), G_3(t, s)$ are nonpositive. If, in addition, $\sum_{j=1}^{p} b_j(t) \chi_{[0, \omega]}(t - \theta_j(t)) \neq 0$, then Green's function $G_4(t, s)$ is also nonpositive.

**Example 4.** Let $r = 1$, $\gamma_1 = 0.8$, $\delta_1 = 0.5$, $t_1 = 0.5$, $\omega = 1$, $p = 1$ and delays satisfy the conditions $t_i \leq t - \tau_i(t) < t_{i+1}$, $t_i \leq t - \theta_i(t) < t_{i+1}$, for $t \in (t_i, t_{i+1})$, $i = 0, ..., r$. Let also $a_1(t) = 0$.

Then $\Omega$ can not exceed 0.5 and $E = 1.6$. Substituting these values into the equation (2) and solving it, we get $\alpha = 4$.

Then the condition (38) can be written in the form:

\[
\underset{t \in [0, \omega]}{\text{esssup}} \sum_{j=1}^{p} |b_j(t)| < 0.8458.
\]
Let us now find an example of a function \( v(t) \) satisfying the condition of Lemma 2. Let us start with \( v(t) = t(m - t) \) in the interval \( t \in [0, t_1) \). The function \( v(t) \) in the rest of the intervals will be of the form

\[
(48) \quad v(t) = v(t_i) + v'(t_i)(t - t_i) - (t - t_i)^2, \quad t \in [t_i, t_{i+1}), \quad i = 1, r, t_{r+1} = \omega,
\]

where the conditions (2) are fulfilled and \( m > 2t_1 \).

Thus,

\[
(49) \quad \begin{cases}
  v(t) = t(m - t), & t \in [0, t_1), \\
  v(t) = v(t_i) + v'(t_i)(t - t_i) - (t - t_i)^2, & t \in [t_i, t_{i+1}), \quad i = 1, r, \quad t_{r+1} = \omega,
\end{cases}
\]

where \( v(t_i) \) and \( v'(t_i) \) can be presented in the forms:

\[
(50) \quad v(t_i) = \begin{cases}
  t_1(m - t_1)\gamma_1, & \text{if } i = 1, \\
  t_1(m - t_1)\prod_{j=1}^{i} \gamma_j + \\
  \sum_{k=2}^{i} v'(t_{k-1})(t_k - t_{k-1})\prod_{j=k}^{i} \gamma_j - \\
  \sum_{k=2}^{i} (t_k - t_{k-1})^2 \prod_{j=k}^{i} \gamma_j, & \text{if } i = 2, r,
\end{cases}
\]

\[
(51) \quad v'(t_i) = \begin{cases}
  (m - 2t_1)\delta_1, & \text{if } i = 1, \\
  (m - 2t_1)\prod_{j=1}^{i} \delta_j - \\
  2\sum_{k=2}^{i} (t_k - t_{k-1})\prod_{j=k}^{i} \delta_j, & \text{if } i = 2, r.
\end{cases}
\]

Let us assume that \( v(t) > 0 \) and substitute this \( v(t) \) into the condition of Lemma 2.

For the next corollary, we use the following notation

\[
(52) \quad \Omega_1 = \max_{i=0,1,\ldots,r} v'(t_i),
\]

\[
(53) \quad \Omega_2 = \max_{i=0,1,\ldots,r+1} v(t_i),
\]

where \( v(t_{r+1}) = v(\omega) \).
Corollary 3. Let $[0, \omega]$ be a semi-nonoscillation interval of (9)-(11). If $a_j(t) \geq 0$, $b_j(t) \geq 0$ for $j = 1, \ldots, p$, $1 \leq \gamma_k$, $1 \leq \delta_k$, $t_k \leq t - \tau_j(t) < t_{k+1}$, $t_k \leq t - \theta_j(t) < t_{k+1}$ for $k = 0, \ldots, r$, $\nu(t)$ defined by (49) is positive for $t \in (0, \omega)$ and

$$
(54) \quad \Omega_1 \sum_{j=1}^{p} a_j(t) + \Omega_2 \sum_{j=1}^{p} b_j(t) < 2,
$$

then Green’s function $G_1(t,s)$ of problem (1)-(3), (4) is nonpositive.

Example 5. Let $r = 1$, $\gamma_1 = \delta_1 = 1.2$, $t_1 = 0.5$, $\omega = 1$. Then the condition (54) can be written in the form:

$$
(55) \quad 1.8333 \sum_{j=1}^{p} a_j(t) + 1.05 \sum_{j=1}^{p} b_j(t) < 2.
$$

REFERENCES


