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ON POSITIVITY OF GREEN'S FUNCTIONS OF TWO-POINT IMPULSIVE PROBLEMS *

IU. MIZGIREVA †

Abstract. We consider the following second order impulsive differential equation with delays

$$\begin{cases}(Lx)(t) \equiv x''(t) + \sum_{j=1}^{p} a_j(t)x'(t-\tau_j(t)) + \sum_{j=1}^{p} b_j(t)x(t-\theta_j(t)) = f(t), \quad t \in [0,\omega], \\ x(t_k) = \gamma_k x(t_k - 0), \quad x'(t_k) = \delta_k x'(t_k - 0), \quad k = 1, 2, ..., r.\end{cases}$$

In this paper we develop the approach of [5] and obtain explicit conditions of nonpositivity of Green's functions for two-point boundary value problems.

Key Words. Second order impulsive differential equations, boundary value problems, sign-constancy of Green's functions

AMS(MOS) subject classification. 34K06, 34K10, 34K45

1. Introduction.

Impulsive differential equations has attracted an attention of a number of recognized mathematicians and has applications in many spheres of science from physics, biology, medicine to economical studies. The following well-known books can be noted in this context [11, 13, 14, 15]. In the book [3] the concept of the general theory of functional differential equations was presented. On the basis of this concept nonoscillation for the first order functional differential equations was considered in [4], where positivity of the Cauchy and Green's functions of the periodic problem was studied. A concept of nonoscillation for the first order differential equations is also considered in the book [1]. The positivity of

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† Department of Mathematics, Ariel University

Green's function of one- and two-point boundary value problems was studied in [2, 5, 6, 7, 8, 9, 10, 12].

This paper develops the approach of [5] and is aimed to obtain explicit conditions of nonpositivity of Green's functions for two-point boundary value problems.

Let us consider the following impulsive equations:

$$(Lx)(t) \equiv x''(t) + \sum_{j=1}^{p} a_j(t)x'(t-\tau_j(t)) + \sum_{j=1}^{p} b_j(t)x(t-\theta_j(t)) = f(t), \quad t \in [0,\omega], \tag{1}$$

$$x(t_k) = \gamma_k x(t_k - 0), \quad x'(t_k) = \delta_k x'(t_k - 0), \quad k = 1, 2, ..., r, \quad 0 = t_0 < t_1 < t_2 < ... < t_r < t_{r+1} = \omega, \tag{2}$$

$$x(\zeta) = 0, \quad \zeta < 0, \tag{3}$$

where f, a_j, b_j: $[0,\omega] \to \mathbb{R}$ are summable functions and τ_j, θ_j: $[0,\omega] \to [0,+\infty)$ are measurable functions for $j = 1, 2, ..., p$, p and r are natural numbers, γ_k and δ_k are real positive numbers.

Let $D(t_1, t_2, ..., t_r)$ be a space of functions x: $[0,\omega] \to \mathbb{R}$ such that their derivative $x'(t)$ is absolutely continuous on every interval $t \in [t_i, t_{i+1})$, $i = 0, 1, ..., r$, $x'' \in L_\infty$ there exist the finite limits $x(t_i - 0) = lim_{t \to t_i^-} x(t)$ and $x'(t_i - 0) = lim_{t \to t_i^-} x'(t)$ and condition (2) is satisfied at points t_i $(i = 0, 1, ..., r)$. Solution x is a function $x \in D(t_1, t_2, ..., t_r)$ satisfying (1)-(3).

2. Preliminaries. For equation (1)-(3) we consider the following variants of boundary conditions:

$$x(0) = 0, \quad x(\omega) = 0, \tag{4}$$

$$x'(0) = 0, \quad x(\omega) = 0, \tag{5}$$

$$x(0) = 0, \quad x'(\omega) = 0, \tag{6}$$

$$x'(0) = 0, \quad x'(\omega) = 0. \tag{7}$$

General solution of the equation (1)-(3) can be represented in the form [4]:

$$x(t) = \nu_1(t)x(0) + C(t,0)x'(0) + \int_0^t C(t,s)f(s)ds, \tag{8}$$

where

- $\nu_1(t)$ is a solution of the homogeneous equation

$$(9)\qquad \begin{aligned}(Lx)(t) \equiv x''(t) &+ \sum_{j=1}^{p} a_j(t)x'(t-\tau_j(t)) \\ &+ \sum_{j=1}^{p} b_j(t)x(t-\theta_j(t)) = 0, \quad t \in [0,\omega],\end{aligned}$$

$$(10)\qquad \begin{aligned}&x(t_k) = \gamma_k x(t_k - 0), \quad x'(t_k) = \delta_k x'(t_k - 0), \quad k = 1,2,...,r, \\ &0 = t_0 < t_1 < t_2 < ... < t_r < t_{r+1} = \omega,\end{aligned}$$

$$(11)\qquad x(\zeta) = 0, \quad \zeta < 0,$$

with the boundary conditions $x(0) = 1$, $x'(0) = 0$.

- $C(t,s)$ is a Cauchy function of the equation (9)-(11).

It means that $C(t,s)$ is the solution of the equation

$$(12)\qquad \begin{aligned}(L_s x)(t) \equiv x''(t) &+ \sum_{j=1}^{p} a_j(t)x'(t-\tau_j(t)) \\ &+ \sum_{j=1}^{p} b_j(t)x(t-\theta_j(t)) = 0, \quad t \in [s,\omega],\end{aligned}$$

$$(13)\qquad \begin{aligned}&x(t_k) = \gamma_k x(t_k - 0), \quad x'(t_k) = \delta_k x'(t_k - 0), \quad k = 1,2,...,r, \\ &0 = t_0 < t_1 < t_2 < ... < t_r < t_{r+1} = \omega,\end{aligned}$$

$$(14)\qquad x(\zeta) = 0, \quad \zeta < s,$$

satisfying the conditions $C(s,s) = 0$, $C'_t(s,s) = 1$, $C(t,s) = 0$ for $t < s$.

If the boundary value problem (1)-(3), (2.i), $i = \overline{1,4}$ is uniquely solvable, then its solution can be represented as

$$(15)\qquad x(t) = \int_0^{\omega} G_i(t,s)f(s)ds, \quad i = \overline{1,4},$$

where $G_i(t,s)$ is Green's function of the problem (1)-(3), (2.i) respectively [5].

Using general representation of the solution (8), the following formulas for Green's functions can be obtained:

$$G_1(t,s) = C(t,s) - C(t,0)\frac{C(\omega,s)}{C(\omega,0)}, \tag{16}$$

$$G_2(t,s) = C(t,s) - C(\omega,s)\frac{\nu_1(t)}{\nu_1(\omega)}, \tag{17}$$

$$G_3(t,s) = C(t,s) - C(t,0)\frac{C'_t(\omega,s)}{C'_t(\omega,0)}, \tag{18}$$

$$G_4(t,s) = C(t,s) - C'_t(\omega,s)\frac{\nu_1(t)}{\nu'_1(\omega)}. \tag{19}$$

Below the following definition will be used.

Definition 1. *We call $[0,\omega]$ a semi-nonoscillation interval of (9)-(11), if every nontrivial solution having zero of derivative does not have zero on this interval.*

In the paper [5] the following assertions about test functions have been proven.

Lemma 1. *Assume that $a_j(t) \geq 0$, $b_j(t) \leq 0$ for $j = 1,...,p$, $0 < \gamma_k \leq 1$, $0 < \delta_k \leq 1$ for $k = 1,...,r$, and there exists a function $v \in D$ and $\epsilon > 0$ such that*

$$(Lv)(t) \geq \epsilon > 0, \quad v(t) > 0, \quad v'(t) < 0, \quad v''(t) > 0, \quad t \in (0,\omega), \tag{20}$$

where the differential operator L is defined by (1). And let $[0,\omega]$ be a semi-nonoscillation interval of (9)-(11). Then Green's functions $G_1(t,s)$, $G_2(t,s)$, $G_3(t,s)$ satisfy the inequalities $G_1(t,s) \leq 0$, $G_2(t,s) \leq 0$, $G_3(t,s) \leq 0$, $(t,s) \in [0,\omega]\times[0,\omega]$. If, in addition, $\sum_{j=1}^{p} b_j(t)\chi_{[0,\omega]}(t-\theta_j(t)) \not\equiv 0$, $t \in [0,\omega]$, then $G_4(t,s) \leq 0$, $(t,s) \in [0,\omega] \times [0,\omega]$.

Lemma 2. *Assume that $a_j(t) \geq 0$, $b_j(t) \geq 0$ for $j = 1,...,p$, $1 \leq \gamma_k$, $1 \leq \delta_k$ for $k = 1,...,r$, and there exists a function $v \in D$ and $\epsilon > 0$ such that*

$$(Lv)(t) \leq -\epsilon < 0, \quad v(t) > 0, \quad v'(t) > 0, \quad v''(t) < 0, \quad t \in (0,\omega), \tag{21}$$

where the differential operator L is defined by (1). And let $[0,\omega]$ be a semi-nonoscillation interval of (9)-(11). Then Green's function $G_1(t,s)$ satisfies the inequality $G_1(t,s) \leq 0$, $(t,s) \in [0,\omega] \times [0,\omega]$.

A question, connected with a particular form of the test functions, satisfying the conditions of Lemma 1 and Lemma 2, will be considered in the next section.

3. Construction of test functions. Let us now find an example of a function $v(t)$ satisfying the condition of Lemma 1. Let us start with $v(t) = e^{-\alpha t}$ in the interval $t \in [0, t_1)$. The function $v(t)$ in the rest of the intervals will be of the form

$$v(t) = c_i e^{-\alpha d_i t}, \quad t \in [t_i, t_{i+1}), \tag{22}$$

q, c_i, $d_i \in \mathbb{R}$, $c_i \neq 0, i = \overline{1,4}$ and the conditions (2) are fulfilled.

After some calculations, we get that $v(t)$ is of the form

$$\begin{cases} v(t) = e^{-\alpha t}, \quad t \in [0, t_1), \\ v(t) = \prod_{j=1}^{i} \gamma_j \, e^{-\alpha \prod_{k=1}^{i} \frac{\delta_k}{\gamma_k}(t-t_i)} \prod_{j=1}^{i-1} e^{-\alpha \prod_{k=1}^{j} \frac{\delta_k}{\gamma_k}(t_{j+1}-t_j)} \, e^{-\alpha t_1}, \\ \qquad\qquad t \in [t_i, t_{i+1}). \end{cases} \tag{23}$$

It should be mentioned that in the paper [5] the form of the function $v(t)$ (see, equation (3.12) from [5]) covers only the case when $\gamma_i = \delta_i$, $i = 1, ..., r$. But our formula (23) is fair for any $\gamma_i > 0$ and $\delta_i > 0$, $i = 1, ..., r$.

For the next theorems, we use the following notation:

$$E = \min_{i=1,2,\ldots,r} \prod_{j=1}^{i} \frac{\delta_j}{\gamma_j}, \tag{24}$$

$$\Theta = \max_{t \in [0,\omega]} \max_{j=1,2,\ldots,p} \theta_j(t), \tag{25}$$

$$T = \max_{t \in [0,\omega]} \max_{j=1,2,\ldots,p} \tau_j(t), \tag{26}$$

$$\Omega = max\{T, \Theta\}. \tag{27}$$

In the theorems below we assume that the delays are small enough to stay within the current interval, i.e. $t - \tau_j \in [t_i, t_{i+1})$ and $t - \theta_j \in [t_i, t_{i+1})$.

Theorem 1. *Let $[0,\omega]$ be a semi-nonoscillation interval of (9)-(11). If $a_j \geq 0$, $b_j \leq 0$, $0 < \gamma_i < \delta_i \leq 1$, $t_i \leq t - \tau_j(t) < t_{i+1}$, $t_i \leq t - \theta_j(t) < t_{i+1}$, for $t \in (t_i, t_{i+1})$, $j = 1, ..., p$, $i = 0, ..., r$, and*

$$\alpha^2 E^2 e^{-\alpha E \Omega} - \alpha E \operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |a_j(t)| > \operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |b_j(t)|, \tag{28}$$

where α satisfies the equation:

$$\alpha(2 - \alpha E\Omega) e^{-\alpha E \Omega} = \frac{1}{E} \operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |a_j(t)|, \tag{29}$$

then Green's functions $G_1(t,s)$, $G_2(t,s)$, $G_3(t,s)$ are nonpositive. If, in addition, $\sum_{j=1}^{p} b_j(t) \chi_{[0,\omega]}(t - \theta_j(t)) \not\equiv 0$, then Green's function $G_4(t,s)$ is also nonpositive.

Proof. Let us substitute this $v(t)$, defined by (23), into the condition of Lemma 1. We obtain:

$$\begin{aligned} \alpha^2 \left(\prod_{j=1}^{i} \frac{\delta_j}{\gamma_j} \right)^2 - \alpha \prod_{j=1}^{i} \frac{\delta_j}{\gamma_j} \operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |a_j(t)| e^{\alpha \prod_{j=1}^{i} \frac{\delta_j}{\gamma_j} \tau_j(t)} - \\ \operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |b_j(t)| e^{\alpha \prod_{j=1}^{i} \frac{\delta_j}{\gamma_j} \theta_j(t)} > 0. \end{aligned} \tag{30}$$

Thus,

$$\alpha^2 E^2 e^{-\alpha E \Omega} - \alpha E \operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |a_j(t)| > \operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |b_j(t)|, \tag{31}$$

where E and Ω are defined by (24) and (27), correspondingly. Denoting

$$F(\alpha) = \alpha^2 E^2 e^{-\alpha E \Omega} - \alpha E \operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |a_j(t)|, \tag{32}$$

we can find its maximum using the derivative:

$$F'(\alpha) = \alpha E^2 (2 - \alpha E \Omega) e^{-\alpha E \Omega} - E \operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |a_j(t)|. \tag{33}$$

The solution of the equation

$$\alpha(2 - \alpha E\Omega) e^{-\alpha E \Omega} = \frac{1}{E} \operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |a_j(t)| \tag{34}$$

will give us a point of maximum.

Using this α in Lemma 1, we obtain a condition of nonpositivity of Green's functions $G_1(t,s)$, $G_2(t,s)$, $G_3(t,s)$. If, in addition, $\sum_{j=1}^{p} b_j(t)\chi_{[0,\omega]}(t-\theta_j(t)) \not\equiv 0$, $t \in [0,\omega]$, then, according to Lemma 1, $G_4(t,s) \leq 0$.

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EXAMPLE 1. *Let $r = 1$, $\gamma_1 = 0.5$, $\delta_1 = 0.8$, $t_1 = 0.5$, $\omega = 1$, $p = 1$ and delays satisfy the conditions $t_i \leq t - \tau_1(t) < t_{i+1}$, $t_i \leq t - \theta_1(t) < t_{i+1}$, for $t \in (t_i, t_{i+1})$, $i = 0, ..., r$. Let also $a_1(t) = 0.3t$.*

Then Ω can not exceed 0.5 and $E = 1.6$. Substituting these values into the equation (29) and solving it, we get $\alpha = 1.9311$.

Then the condition (28) can be written in the form:

$$\operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |b_j(t)| < 1.1097. \tag{35}$$

If $a_j(t)$ is close to zero for $t \in [0,\omega]$, $j = 1, ..., p$, then the following Corollary is fulfilled.

COROLLARY 1. *Let $[0,\omega]$ be a semi-nonoscillation interval of (9)-(11). If $a_j \approx 0$, $b_j \leq 0$, $0 < \gamma_i < \delta_i \leq 1$, $t_i \leq t - \tau_j(t) < t_{i+1}$, $t_i \leq t - \theta_j(t) < t_{i+1}$, for $t \in (t_i, t_{i+1})$, $j = 1, ..., p$, $i = 0, ..., r$, and*

$$\frac{4}{\Omega^2} e^{-2} > \operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |b_j(t)|, \tag{36}$$

then Green's functions $G_1(t,s)$, $G_2(t,s)$, $G_3(t,s)$ are nonpositive. If, in addition, $\sum_{j=1}^{p} b_j(t)\chi_{[0,\omega]}(t-\theta_j(t)) \not\equiv 0$, then Green's function $G_4(t,s)$ is also nonpositive.

EXAMPLE 2. *Let $r = 1$, $\gamma_1 = 0.5$, $\delta_1 = 0.8$, $t_1 = 0.5$, $\omega = 1$, $p = 1$ and delays satisfy the conditions $t_i \leq t - \tau_1(t) < t_{i+1}$, $t_i \leq t - \theta_1(t) < t_{i+1}$, for $t \in (t_i, t_{i+1})$, $i = 0, ..., r$. Let also $a_1(t) = 0$.*

Then Ω can not exceed 0.5 and $E = 1.6$. Substituting these values into the equation (29) and solving it, we get $\alpha = 2.5$.

Then the condition (28) can be written in the form:

$$\operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |b_j(t)| < 2.1653. \tag{37}$$

Theorem 2. *Let $[0,\omega]$ be a semi-nonoscillation interval of (9)-(11). If $a_j \geq 0$, $b_j \leq 0$, $0 < \delta_i \leq \gamma_i \leq 1$, $t_i \leq t - \tau_j(t) \leq t_{i+1}$, $t_i \leq t - \theta_j(t) \leq t_{i+1}$, for $t \in (t_i, t_{i+1})$, $j = 1, ..., p$, $i = 0, ..., r$, and*

$$\alpha^2 E^2 e^{-\alpha\Omega} - \alpha E \operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |a_j(t)| > \operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |b_j(t)|, \tag{38}$$

where α satisfies the equation:

$$\alpha(2-\alpha\Omega)e^{-\alpha\Omega} = \frac{1}{E}\operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |a_j(t)|, \tag{39}$$

then Green's functions $G_1(t,s)$, $G_2(t,s)$, $G_3(t,s)$ are nonpositive. If, in addition, $\sum_{j=1}^{p} b_j(t)\chi_{[0,\omega]}(t-\theta_j(t)) \not\equiv 0$, then Green's function $G_4(t,s)$ is also nonpositive.

Proof. Let us substitute this $v(t)$, defined by (23), into the condition of Lemma 1. We obtain:

$$\begin{aligned} &\alpha^2 \left(\prod_{j=1}^{i} \frac{\delta_j}{\gamma_j}\right)^2 - \alpha \prod_{j=1}^{i} \frac{\delta_j}{\gamma_j} \operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |a_j(t)| e^{\alpha\tau_j(t)} - \\ &\qquad\qquad \operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |b_j(t)| e^{\alpha\theta_j(t)} > 0. \end{aligned} \tag{40}$$

Thus,

$$\alpha^2 E^2 e^{-\alpha\Omega} - \alpha E \operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |a_j(t)| > \operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |b_j(t)|, \tag{41}$$

where E and Ω are defined by (24) and (27), correspondingly. Denoting

$$F(\alpha) = \alpha^2 E^2 e^{-\alpha\Omega} - \alpha E \operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |a_j(t)|, \tag{42}$$

we can find its maximum using the derivative:

$$F'(\alpha) = \alpha E^2 (2-\alpha\Omega) e^{-\alpha\Omega} - E \operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |a_j(t)|. \tag{43}$$

The solution of the equation

$$\alpha(2-\alpha\Omega)e^{-\alpha\Omega} = \frac{1}{E}\operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |a_j(t)| \tag{44}$$

will give us a point of maximum.

Using this α in Lemma 1, we obtain a condition of nonpositivity of Green's functions $G_1(t,s)$, $G_2(t,s)$, $G_3(t,s)$. If, in addition, $\sum_{j=1}^{p} b_j(t)\chi_{[0,\omega]}(t-\theta_j(t)) \not\equiv 0$, $t \in [0,\omega]$, then, according to Lemma 1, $G_4(t,s) \le 0$.

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Example 3. *Let $r=1$, $\gamma_1 = 0.8$, $\delta_1 = 0.5$, $t_1 = 0.5$, $\omega = 1$, $p = 1$ and delays satisfy the conditions $t_i \le t - \tau_1(t) < t_{i+1}$, $t_i \le t - \theta_1(t) < t_{i+1}$, for $t \in (t_i, t_{i+1})$, $i = 0, ..., r$. Let also $a_1(t) = 0.3t$.*

Then Ω can not exceed 0.5 and $E = 1.6$. Substituting these values into the equation (2) and solving it, we get $\alpha = 2.6389$.

Then the condition (38) can be written in the form:

$$\operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |b_j(t)| < 0.2322. \tag{45}$$

If $a_j(t)$ is close to zero for $t \in [0,\omega]$, $j = 1, ..., p$, then the following Corollary is fulfilled.

Corollary 2. *Let $[0,\omega]$ be a semi-nonoscillation interval of (9)-(11). If $a_j \approx 0$, $b_j \le 0$, $0 < \delta_i \le \gamma_i \le 1$, $t_i \le t - \tau_j(t) \le t_{i+1}$, $t_i \le t - \theta_j(t) \le t_{i+1}$, for $t \in (t_i, t_{i+1})$, $j = 1, ..., p$, $i = 0, ..., r$, and*

$$\frac{4E^2}{\Omega^2} e^{-2} > \operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |b_j(t)|, \tag{46}$$

then Green's functions $G_1(t,s)$, $G_2(t,s)$, $G_3(t,s)$ are nonpositive. If, in addition, $\sum_{j=1}^{p} b_j(t)\chi_{[0,\omega]}(t-\theta_j(t)) \not\equiv 0$, then Green's function $G_4(t,s)$ is also nonpositive.

Example 4. *Let $r=1$, $\gamma_1 = 0.8$, $\delta_1 = 0.5$, $t_1 = 0.5$, $\omega = 1$, $p = 1$ and delays satisfy the conditions $t_i \le t - \tau_1(t) < t_{i+1}$, $t_i \le t - \theta_1(t) < t_{i+1}$, for $t \in (t_i, t_{i+1})$, $i = 0, ..., r$. Let also $a_1(t) = 0$.*

Then Ω can not exceed 0.5 and $E = 1.6$. Substituting these values into the equation (2) and solving it, we get $\alpha = 4$.

Then the condition (38) can be written in the form:

$$\operatorname*{esssup}_{t\in[0,\omega]} \sum_{j=1}^{p} |b_j(t)| < 0.8458. \tag{47}$$

Let us now find an example of a function $v(t)$ satisfying the condition of Lemma 2. Let us start with $v(t) = t(m-t)$ in the interval $t \in [0, t_1)$. The function $v(t)$ in the rest of the intervals will be of the form

$$v(t) = v(t_i) + v'(t_i)(t-t_i) - (t-t_i)^2,\ t \in [t_i, t_{i+1}),\ i = \overline{1,r},\ t_{r+1} = \omega, \tag{48}$$

where the conditions (2) are fulfilled and and $m > 2t_1$.

Thus,

$$\begin{cases} v(t) = t(m-t), & t \in [0, t_1), \\ v(t) = v(t_i) + v'(t_i)(t-t_i) - (t-t_i)^2, & t \in [t_i, t_{i+1}), \\ & \quad i = \overline{1,r}, \quad t_{r+1} = \omega, \end{cases} \tag{49}$$

where $v(t_i)$ and $v'(t_i)$ can be presented in the forms:

$$v(t_i) = \begin{cases} t_1(m-t_1)\gamma_1, & \texttt{if} \quad i = 1, \\ t_1(m-t_1)\prod_{j=1}^{i}\gamma_j + \\ \qquad \sum_{k=2}^{i}\left[v'(t_{k-1})(t_k - t_{k-1})\prod_{j=k}^{i}\gamma_j\right] - \\ \qquad\qquad \sum_{k=2}^{i}\left[(t_k - t_{k-1})^2\prod_{j=k}^{i}\gamma_j\right], & \texttt{if} \quad i = \overline{2,r}, \end{cases} \tag{50}$$

$$v'(t_i) = \begin{cases} (m-2t_1)\delta_1, & \texttt{if} \quad i = 1, \\ (m-2t_1)\prod_{j=1}^{i}\delta_j - \\ \qquad 2\sum_{k=2}^{i}\left[(t_k - t_{k-1})\prod_{j=k}^{i}\delta_j\right], & \texttt{if} \quad i = \overline{2,r}. \end{cases} \tag{51}$$

Let us assume that $v(t) > 0$ and substitute this $v(t)$ into the condition of Lemma 2.

For the next collorary, we use the following notation

$$\Omega_1 = \max_{i=0,1,\ldots,r} v'(t_i), \tag{52}$$

$$\Omega_2 = \max_{i=0,1,\ldots,r+1} v(t_i), \tag{53}$$

where $v(t_{r+1}) = v(\omega)$.

COROLLARY 3. *Let* $[0, \omega]$ *be a semi-nonoscillation interval of (9)-(11). If* $a_j(t) \geq 0$, $b_j(t) \geq 0$ *for* $j = 1, ..., p$, $1 \leq \gamma_k$, $1 \leq \delta_k$, $t_k \leq t - \tau_j(t) < t_{k+1}$, $t_k \leq t - \theta_j(t) < t_{k+1}$ *for* $k = 0, ..., r$, $v(t)$ *defined by (49) is positive for* $t \in (0, \omega)$ *and*

$$\Omega_1 \sum_{j=1}^{p} a_j(t) + \Omega_2 \sum_{j=1}^{p} b_j(t) < 2, \tag{54}$$

then Green's function $G_1(t, s)$ *of problem (1)-(3), (4) is nonpositive.*

EXAMPLE 5. *Let* $r = 1$, $\gamma_1 = \delta_1 = 1.2$, $t_1 = 0.5$, $\omega = 1$. *Then the condition (54) can be written in the form:*

$$1.8333 \sum_{j=1}^{p} a_j(t) + 1.05 \sum_{j=1}^{p} b_j(t) < 2. \tag{55}$$

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